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On Homomorphism-homogeneous and Polymorphism-homogeneous S-metric Spaces

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Contents

1 Introduction

Structures represent the staple elements in many branches of mathematics, particularly in model theory. One way of developing a better understanding of some class of structures, is by working on their classification in accordance with some specific property. Among such is the classical notion of homogeneity (the property of those relational structures for which every local isomorphism can be extended to an automorphism of the whole structure) which was brought to light by Roland Fraïssé in his exceptional works from 1954. There, he considered a suitable set of finite structures starting from which he managed to build their 'limit'. The latter proved itself to be remarkably interesting, leading to the development of a rich theory about the constructions of homogeneous structures in general [10]. As a result that sparked an interest so deep that a fairly natural generalisation of homogeneity, in the form of homomorphism-homogeneity followed in 2002, provided by Cameron and Nešetřil [4]. A structure is called homomorphism-homogeneous in case every local homomorphism may be extended to an endomorphism of the structure in point. This property was then followed by yet another stronger modification in 2015, namely the polymorphism-homogeneity. Broadly speaking it is a somewhat more tangible property, introduced by Pech and Pech in [15], where, in a sense, polymorphisms take over.

The whole thesis revolved around exploring the homomorphism-homogeneous and polymorphism-homogeneous metric spaces. What is understood under the term S-metric space, for $S \subseteq [0, \infty)$, is the class of all metric spaces with distances taking values in S. In 2007 Delhommé, Laflamme, Pouzet, and Sauer gave a complete characterisation of homogeneous S-metric spaces with respect to S [6]. For example, if $S = [0, \infty)$ the corresponding structure is the Urysohn space (the Fraïssé limit of the class of all finite metric spaces with rational distances). These earlier results were the ones which prompted us to initially commence our research on S-metric spaces. Nevertheless, progress made us realise that this reference to the set of values of the distances within a specific metric space was superfluous for our approach. Clearly, every metric space on its own is an S-metric space for an adequately chosen S. Therefore, there shall be no further reference to S-metric spaces, but simply metric spaces satisfying some specific properties, in the rest of this thesis.

From the very beginning we were perfectly aware that a full classification of metric spaces with respect to homomorphism-homogeneity was highly unlikely, to say the least. This is due to the following Theorem by Rusinov and Schweitzer [21], as shell be explained in full detail, later on.

Theorem 1.1 *Deciding whether a finite graph with loops is homomorphism-homogeneous is a* **coNP***-complete problem.*

On the other hand, it made us wonder whether polymorphism-homogenous metric spaces would be nearly as difficult to tackle. That was our starting point which eventually gave us a glimpse into the classification of **finite** and **countably infinite** polymorphism-homogenous metric spaces. Interestingly, the full classification of finite metrically polymorphism-homogenous connected graphs up to diameter 3 came as an extra result.

Finally, I take this opportunity to express my deepest gratitude to my supervisor Professor Maja Pech for the numerous invaluable pieces of advice she gave me, for all the support and constructive criticism, and not to forget for all the patience and generous amount of time she has spent rereading and correcting my drafts. I am also greatly indebted to Professor Dragan Mašulović, who played a crucial role in the very choice of the topic for this thesis, for his continual considerable support and great collaboration throughout my studies.

2 Preliminaries

2.1 Overall Notation

The set of all non-negative integers, shall be denoted by $\mathbb{N}$, whereas $\mathbb{N}_+ := \mathbb{N}\backslash\{0\}$. For any set A, $|A|$ represents the *cardinality* of A. The *product* of the family of sets $(A_i)_{i\in I}$ is defined by

$$\prod_{i\in I} A_i := \{(a_i)_{i\in I} | \forall i \in I : a_i \in A_i\}.$$

When all A_i are equal to A, for each $i \in I$, then we shorthand the product $\prod_{i\in I} A_i$ for A^I, and refer to it as the *direct power* of A. For the special case of $I = k$, where k is a nonzero natural number, we make the identification $k = \{0, 1, \dots, k-1\}$, and call A^k the k^{th} *power* of A. In order to simplify the notation, all of its elements shall be overlined. More over, the i^{th} component of any $\bar{x} \in A^k$ will be x_i, for $i \in \{1, 2, \dots, k\}$, unless stated otherwise. In other words, every element of A^k is by default considered to be of the form $\bar{x} = (x_1, x_2, \dots, x_k)$, for $k \in \mathbb{N}_+$.

Furthermore, let A and B be such that $f : A \to B$ is a mapping from A to B. Then we say that A is the *domain* of f, in symbols $\text{dom}(f) := A$. The *image set* of f, denoted by $\text{im}(f)$, is of course the set of all $f(x)$'s for $x \in \text{dom}(f)$. For some $C \subseteq A$ we say that $f_{\restriction C} : C \to B : y \mapsto f(y)$ is the *restriction* of f to C. A function g is an *extension* of f if $g \supseteq f$, i.e. $\text{dom}(f) \subseteq \text{dom}(g)$ and $g(x) = f(x)$ for all $x \in \text{dom}(f)$. For any $D \subseteq A$ *the image of D under f* is:

$$f[D] := \{y \in B : (\exists x \in D) f(x) = y\}.$$

An *n-ary operation* on A is a function $f : A^n \to A$, where $n \in \mathbb{N}_+$. Then $f(\bar{a}) := (f(a_1), \ldots, f(a_n))$, where the n-tuple $\bar{a} \in A^n$. However, when f is a function with n arguments on A, meaning that $\text{dom}(f) = A^n$, for any $a_1, \ldots, a_n \in A$ we still write $f(a_1, \ldots, a_n) =: f(\bar{a})$ having defined $\bar{a} := (a_1, \ldots, a_n)$. This exploitation of the same notation is legitimate as it shall cause no ambiguity further on. It shall always be perfectly clear from the context which one of these two uses we have in mind.

In the following two subsections, upon introducing some basic notions from model and graph theory, we will be heavily relying on [10], [15] and [11]. Additional references will be given, where appropriate.

2.2 Basic model theoretical notions

Representing one of the most fundamental notions in mathematics in general it does not come as a surprise that structures alone are the main subject-matter of model theory and thus require a formal introduction which is closely intertwined with the concept of a signature.

Definition 2.1 A *signature* L is a triple (F, R, C, ar), where:

- F is a set of function symbols,
- R is a set of relation symbols,
- C is a set of constant symbols and
- $ar : F \cup R \to \mathbb{N}_+$.

Aside from that the three sets F, R and C are assumed to be pairwise disjoint.

Signatures should be chosen in such a way that the notions of homomorphism and substructure agree with the usual notions for the relevant branch of mathematics. Now we have:

Definition 2.2 An *L-structure* $\mathbf{A}$ is a tuple $(A, (f^{\mathbf{A}})_{f\in F}, (\rho^{\mathbf{A}})_{\rho\in R}, (c^{\mathbf{A}})_{c\in C})$ where

- A is a set;
- $f^{\mathbf{A}} : A^{ar(f)} \to A$, for each $f \in F$;
- $\rho^{\mathbf{A}} \subseteq A^{ar(\rho)}$, for each $\rho \in R$;
- $c^{\mathbf{A}} \in A$, for each $c \in C$.

The set A is often referred to as the *universe* or *carrier* of $\mathbf{A}$.

A signature L with no constants or function symbols is called a *relational signature*, and such an L-structure is said to be a *relational structure*.

With one "grand sweep of the arm" we get to fix the term *homomorphism* in basically any branch of mathematics, a term of crucial importance for our further research.

Definition 2.3 Let L be a signature and let $\mathbf{A}$ and $\mathbf{B}$ be L-structures. By a *homomorphism* h from $\mathbf{A}$ to $\mathbf{B}$, in symbols $h : \mathbf{A} \to \mathbf{B}$, we shall mean a mapping h from A to B which satisfies the following three properties.

(1) For each constant symbol c of L, $h(c^{\mathbf{A}}) = c^{\mathbf{B}}$.

(2) For each relation symbol ρ of L and a tuple $\bar{a} = (a_1, \dots, a_{ar(\rho)})$ from A, if $\bar{a} \in \rho^{\mathbf{A}}$ then $h(\bar{a}) \in \rho^{\mathbf{B}}$.

(3) For each function symbol f of L and a tuple $\bar{a} = (a_1, \dots, a_{ar(f)})$ from A, $h(f_{\mathbf{A}}(\bar{a})) = f_{\mathbf{B}}(h(\bar{a}))$.

By an *embedding* of $\mathbf{A}$ into $\mathbf{B}$ we mean a homomorphism $h : \mathbf{A} \to \mathbf{B}$ which is injective and satisfies the following stronger version of (2):

> For each relation symbol ρ of L and a tuple $\bar{a} := (a_1, \dots, a_{ar(\rho)}) \in A^{ar(\rho)}$, $\bar{a} \in \rho^{\mathbf{A}}$ if and only if $h(\bar{a}) \in \rho^{\mathbf{B}}$.

We say that $\mathbf{B}$ is a *substructure* of $\mathbf{A}$ (and write $\mathbf{B} \leqslant \mathbf{A}$) if $B \subseteq A$ and the inclusion map $\iota : B \to A$ is an embedding. One should bear in mind that substructures are fully determined by their universe. A homomorphism $h : \mathbf{D} \to \mathbf{B}$ is called a *partial homomorphism* from $\mathbf{A}$ to $\mathbf{B}$ with the domain $\mathbf{D} \leqslant \mathbf{A}$ (written as $h : \mathbf{A} \multimap\!\!\to \mathbf{B}$). The structure $\mathbf{D}$ will usually be denoted by $\underline{\operatorname{dom} f}$. Let $\mathbf{C}$ be an L-structure and X a set of elements of $\mathbf{C}$. The unique smallest substructure $\mathbf{D}$ of $\mathbf{C}$ whose domain includes X is called *the* *substructure of* $\mathbf{C}$ *generated by* X, in symbols $\mathbf{D} = \mathbf{C}[X]$. We call X a *set of generators* for $\mathbf{D}$. A structure $\mathbf{C}$ is said to be *finitely generated* if $\mathbf{C}$ is of the form $\mathbf{C}[X]$ for some **finite** set X of its elements. Now, a homomorphism $h : \mathbf{A} \to \mathbf{A}$ is called an *endomorphism* of $\mathbf{A}$, whereas a partial homomorphism of a structure to itself is called a *partial endomorphism*. Further, a *local homomorphism of* $\mathbf{A}$ is a homomorphism from a finitely generated substructure of $\mathbf{A}$ to $\mathbf{A}$ and an *epimorphism* is a surjective homomorphism. Additionally, an *isomorphism* is a surjective embedding, and an isomorphism $h : \mathbf{A} \to \mathbf{A}$ is called an *automorphism* of $\mathbf{A}$. The set of all endomorphisms of $\mathbf{A}$ shall, traditionally, be denoted by $\mathrm{End}(\mathbf{A})$, whereas $\mathrm{Aut}(\mathbf{A})$ stands for the set of all automorphisms of $\mathbf{A}$. Two L-structures $\mathbf{A}$ and $\mathbf{B}$ are *isomorphic*, in symbols $\mathbf{A} \cong \mathbf{B}$, in case there exists an isomorphism between these two structures.

2.3 Basic graph theoretical notions

Throughout the paper, a graph is considered to be a simple graph with all the loops included. In particular, a graph $\mathbf{G}$ is a pair $(V(\mathbf{G}), E(\mathbf{G}))$, where $V(\mathbf{G})$ is a set of vertices and the set of edges $E(\mathbf{G})$ is a symmetric, reflexive binary relation on $V(\mathbf{G})$. In particular, K_n represents the *complete graph* on n vertices. By $n \cdot \mathbf{G}$ we denote the disjoint union of n copies of $\mathbf{G}$. Let $u, v \in V(\mathbf{G})$. The *distance* between u and v in $\mathbf{G}$ is the length of the shortest existing path connecting them and is denoted by $d_{\mathbf{G}}(u, v)$. In case u and v are disconnected, then the distance between them is infinite. Moreover, $\varepsilon(u) := \max\limits_{v \in V(\mathbf{G})} d_{\mathbf{G}}(u, v)$ represents the *eccentricity* of a vertex $u \in V(\mathbf{G})$ in $\mathbf{G}$. The *diameter* of $\mathbf{G}$, denoted by $\mathrm{diam}(\mathbf{G})$, is the greatest distance between any pair of vertices in it, whereas $r(\mathbf{G}) := \min\limits_{u \in V(\mathbf{G})} \varepsilon(u)$ stands for the *radius* of $\mathbf{G}$. In case of a disconnected graph the diameter and radius of it are infinite. In general, it holds:

$$r(\mathbf{G}) \leqslant \mathrm{diam}(\mathbf{G}) \leqslant 2r(\mathbf{G}).$$

A vertex connected to all the vertices of the graph which contains it is called a *universal vertex* [21].

Remark. In a graph $\mathbf{G}$ of radius $\geqslant 3$ there exists no universal vertex. The eccentricity of one such would otherwise be 1, implying that $r(\mathbf{G}) = 1$ and the diameter at most twice as much, i.e. 2.

An isomorphism between the graphs $\mathbf{G}$ and $\mathbf{H}$ is a bijective mapping $f : V(\mathbf{G}) \to V(\mathbf{H})$ such that $(x, y) \in E(\mathbf{G})$ if and only if $(f(x), f(y)) \in E(\mathbf{G})$, for all $x, y \in V(\mathbf{G})$. By $\mathbf{G} \cong \mathbf{H}$ we denote that $\mathbf{G}$ and $\mathbf{H}$ are *isomorphic*.

A *homomorphism* between the graphs $\mathbf{G}$ and $\mathbf{H}$ is a mapping $f : V(\mathbf{G}) \to V(\mathbf{H})$ such that $(x, y) \in E(\mathbf{G})$ implies $(f(x), f(y)) \in E(\mathbf{G})$, for all $x, y \in V(\mathbf{G})$. An *endomorphism* of $\mathbf{G}$ is a homomorphism from $\mathbf{G}$ into itself. Since a graph can be seen as a relational structure, the term *subgraph* of $\mathbf{G}$ will always refer to the substructure of $\mathbf{G}$ in the model theoretical sense. Take $f \in \mathrm{End}(\mathbf{G})$ and $S \subseteq V(\mathbf{G})$. If $\mathbf{G}[S]$ is connected, then so is $\mathbf{G}[f(S)]$; notably, f maps a connected component into a connected component.

2.4 Homogeneity and its generalisations

From now on, we shall assume that our signature is always a relational one, unless explicitly stated otherwise. Thus, a signature L is of the form $(\emptyset, R, \emptyset, ar)$, or shorter $L = (R, ar)$. This restriction has the advantage that finite structures are precisely the same as finitely generated structures.

Definition 2.4 A relational structure is called *homogeneous* if every isomorphism between finite substructures extends to an automorphism of the given structure.

This term found itself a generalisation, once the isomorphisms were "substituted" for homomorphisms.

Definition 2.5 A relational structure is *homomorphism-homogeneous* if every homomorphism between finite substructures extends to an endomorphism of the very structure.

Further on, in order to introduce *polymorphisms* we shall require the notion of the power of a (relational) structure. Hence, firstly, let I be a set and, for $i \in I$, let $\mathbf{A}_i = (A_i, (\rho^{\mathbf{A}_i})_{\rho \in L})$ be a relational structure. The *product* of the family $(\mathbf{A}_i)_{i \in I}$ is defined by

$$\mathbf{A} = \prod_{i \in I} \mathbf{A}_i := \left(\prod_{i \in I} A_i, (\rho^{\mathbf{A}})_{\rho \in L} \right)$$

whereas for all $\rho \in L$, we have

$$\rho^{\mathbf{A}} := \{((a_{1,i})_{i \in I}, \ldots, (a_{ar(\rho),i})_{i \in I}) | \forall i \in I : (a_{1,i}, \ldots, a_{ar(\rho),i}) \in \rho^{\mathbf{A}_i}\}.$$

When all $\mathbf{A}_i$ are equal to one and the same structure $\mathbf{B}$, then we abbreviate the product $\prod_{i \in I} \mathbf{A}_i$ by $\mathbf{B}^I$. This special kind of direct product is called a *direct power* of $\mathbf{B}$.

If $\emptyset \neq J \subseteq I$, we denote the *projection homomorphism* with respect to J by

$$\pi_J : \prod_{i \in I} \mathbf{A}_i \to \prod_{i \in J} \mathbf{A}_i, \ \text{ where } \ (a_i)_{i \in I} \mapsto (a_i)_{i \in J}.$$

In the special case when $J = \{j\}$, we write π_j instead of $\pi_{\{j\}}$ and call it the $j^t h$ *projection* (homomorphism.)

Let L be a relational signature and $\mathbf{A} = (A, (\rho^{\mathbf{A}})_{\rho \in L})$ be an L-structure. Then the *k-ary polymorphisms* of $\mathbf{A}$ are defined to be the homomorphisms from $\mathbf{A}^k$ to $\mathbf{A}$. *Partial* and *local k-ary polymorphisms* of $\mathbf{A}$ are defined accordingly, as partial or local homomorphisms from $\mathbf{A}^k$ to $\mathbf{A}$, respectively. The set of all polymorphisms of $\mathbf{A}$ will be denoted by $\mathrm{Pol}(\mathbf{A})$, while the set of all k-ary polymorphisms will be denoted by $\mathrm{Pol}^{(k)}(\mathbf{A})$. Notice that the unary polymorphisms of $\mathbf{A}$ are in fact endomorphisms of $\mathbf{A}$. It is now fairly evident, from the above most broad definition of homomorphisms, that partial k-ary polymorphisms of $\mathbf{A}$ are characterised by the following property:

> A partial function $f : A^k \multimap\!\!\rightarrow A$ is a partial polymorphism of $\mathbf{A}$ if and only if for all $\rho \in L$ and for all $\bar{a}_1, \ldots, \bar{a}_{ar(\rho)} \in \operatorname{dom} f$ with $\bar{a}_i = (a_{i,1}, \ldots, a_{i,k})$, we have

$$\begin{bmatrix} a_{1,1} \\ \vdots \\ a_{ar(\rho),1} \end{bmatrix} \in \rho^{\mathbf{A}}, \ldots, \begin{bmatrix} a_{1,k} \\ \vdots \\ a_{ar(\rho),k} \end{bmatrix} \in \rho^{\mathbf{A}} \Longrightarrow \begin{bmatrix} f(a_{1,1}, \ldots, a_{1,k}) \\ \vdots \\ f(a_{ar(\rho),1}, \ldots, a_{ar(\rho),k}) \end{bmatrix} \in \rho^{\mathbf{A}}.$$

We say that a relational structure is *k-polymorphism-homogeneous* if every k-ary local polymorphism of $\mathbf{A}$ can be extended to a polymorphism of $\mathbf{A}$. If $\mathbf{A}$ is k-ary polymorphism-homogeneous for every $k \in \mathbb{N}_+$, then we say that $\mathbf{A}$ is *polymorphism-homogeneous*. In other words:

Definition 2.6 A relational structure is called *polymorphism-homogeneous* if every partial polymorphism with finite domain extends to a global polymorphism of the structure.

Remark. Out of convenience, we may, at times, abbreviate the term homomorphism-homogeneous to HH, and similarly the term polymorphism-homogeneous to PH, throughout the text. Notice, then how if a structure is 1-PH then it is actually, HH. Thus, by definition, every PH structure is at the same time an HH one.

The key correlation between the properties of polymorphism-homogeneity and homomorphism-homogeneity, being also the one which will be used time and time again, is the one presented below:

Proposition 2.7 ([15]) *A structure* $\mathbf{A}$ *is polymorphism-homogeneous if and only if* $\mathbf{A}^k$ *is homomorphism-homogeneous, for every* $k \in \mathbb{N}_+$.

Additionally, we shall also require one more structural property, alongside with its application:

Definition 2.8 We say that a relational structure $\mathbf{A}$ has the *one-point extension property* if for every finite substructure $\mathbf{B}$ of $\mathbf{A}$, every $b \in A \backslash B$ and every homomorphism $f : \mathbf{B} \to \mathbf{A}$, there exists a homomorphism $g : \mathbf{B} \cup \{b\} \to \mathbf{A}$ which extends f.

Proposition 2.9 ([16]) *If a finite or countably infinite relational structure* $\mathbf{A}$ *has the one-point extension property, then it is homomorphism-homogeneous.*

3 Polymorphism-homogeneity

The phenomenon of polymorphism-homogeneity, which was formally first fixed in [15], did in fact appear in a number of papers beforehand, though admittedly not in the exact same context (one can gain insight from this very paper). For instance, one may consider the Baker-Pixley Theorem [1] in universal algebra, to represent its first notable occurrence, a powerful tool generalising the Chinese remainder theorem and Langrange's interpolation theorem. Similarly, in [19], motivated by questions from multivalued logics and clone theory, Romov studied this property, even extending his approach to countably infinite structures [18, 20]. Another source comes from Kaarli [12], where he deals with meet-complete lattices of equivalence relations, and owing to this characterisation he manages to identify classes of locally affine complete algebras.

The interpolation condition (IC), which is an instrumental part of the theory of natural dualities [5] is, also, in some way related to polymorphism-homogeneity. In fact, a structure has the (IC) if every partial (not necessarily local) polymorphism extends to a global one. In a nutshell, structures that have the (IC) are actually polymorphism-homogeneous.

The diversity of contexts in which this property can be found motivated the authors of [15] to go one step further, and look for other, related, model-theoretic notions such as quantifier elimination, extending and generalising earlier results by Romov [18]. Moreover, they derived an algebraic characterisation of the structures that fulfill the interpolation condition among all polymorphism-homogeneous structures. One more significant result coming from their paper is the claim of decidability of polymorphism-homogeneity for finite structures, which is shown there. The most recent papers on the classification of polymorphism-homogeneous structures include the one on monounary algebras by Farkasová and Jakubíková-Studenovská [8] and the one on finite tournaments with loops by Feller [9].

One rather direct approach for uncovering more of the PH classes of structures consists of examining the known classes of HH structures and determining which of those possess this finer property. It did not come as a surprise that upon imposing this stricter condition the number of remaining structures in a certain class should drastically decrease. One straightforward example is provided below.

Example 3.1 As explained above, in order to establish which finite irreflexive binary relational systems are polymorphism-homogeneous, it suffices to look no further but to [14]. From that paper we obtain the list of our potential candidates, which are isomorphic to exactly one of the following:

(1) $k \cdot K_n$ for some $k, n \geqslant 1$;

(2) $k \cdot C_3$ for some $k \geqslant 1$, where C_3 denotes the oriented 3-cycle.

From [15] we gather that a finite graph of the form $\mathbf{A} = (A, \rho^{\mathbf{A}})$ is polymorphism-homogeneous if and only if either all of its connected components are isomorphic to K_1 (in which case we have $\rho^{\mathbf{A}} = \emptyset$) or each connected component of $\mathbf{A}$ is isomorphic to K_2. On the other hand, according to [9] we have that

$$(k \cdot C_3)^n \cong l \cdot C_3,$$

where $l := 3^{n-1}k^n$ and n a positive integer. With that in mind, any positive power of k disjoint copies of C_3 is trivially homomorphism-homogeneous. As a result, applying Proposition 2.7, $k \cdot C_3$ is polymorphism-homogeneous for any $k \in \mathbb{N}_+$.

Finally, we can list the finite irreflexive polymorphism-homogeneous binary relational systems as below:

(1) $k \cdot K_n$ for some $k \geqslant 1$ and $n \in \{1, 2\}$;

(2) $k \cdot C_3$ for some $k \geqslant 1$. □

Up until 2009, the classification of homomorphism-homogeneous structures was more of a combinatorial problem driven by sheer curiosity and mathematical enthusiasm. Classes of structures were considered on their own merits and each and every time either a new method had to be derived or an old one needed some adjusting. However, in [16] a general criterion for the classification of homomorphism-homogeneous relational structures was eventually developed by M. Pech, with the introduction of *witnesses* — special configurations which are forbidden in all the relational structures that have the one-point extension property, or in other words in the ones which are homomorphism-homogeneous. More formally:

Definition 3.1 Given a relational structure $\mathbf{A} = (A, (\rho^{\mathbf{A}})_{\rho \in R})$ and its finite substructure $\mathbf{B} = (B, (\rho^{\mathbf{B}})_{\rho \in R})$ we say that $c \in A$ is a *weak centre* of $\mathbf{B}$ if for every $b \in B$ there exists a $\rho \in R$, $b_3, \ldots, b_{ar(\rho)} \in B$ and an $\alpha \in Sym\{1, 2, \ldots, ar(\rho)\}$ such that

$$(c, b, b_3, \ldots, b_{ar(\rho)})^{\alpha} \in \rho^{\mathbf{A}}.$$

Definition 3.2 A *witness* is a quadruple $(\mathbf{B}_1, \mathbf{B}_2, f, c)$, such that $\mathbf{B}_1$ is a finite substructure of $\mathbf{A}$, c is a weak center of $\mathbf{B}_1$ in $\mathbf{A}$, $\mathbf{B}_2$ is a substructure of $\mathbf{A}$, and $f : \mathbf{B}_1 \twoheadrightarrow \mathbf{B}_2$ is surjective, but f cannot be extended to $\mathbf{B}_1 \cup \{c\}$.

What this approach provided was, practically speaking, a unique tool for determining when a given relational structure is homomorphism-homogeneous. Since one may observe polymorphisms as merely a special kind of homomorphisms, due to Proposition 2.7, what immediately springs to mind is to eliminate those structures whose squares are no longer homomorphism-homogeneous upon spotting a witness. The next two examples are a case in point.

Example 3.2 There is, up to isomorphism, just one connected noncentral homomorphism-homogenous tolerance relation on 4 vertices. It may graphically be represented as the bipartite graph $\mathbf{A} = (A, \varrho^{\mathbf{A}})$ from Figure 1. It is also a special case of an infinite family of homomorphism-homogeneous tolerance relations described in [16] (3.2.5. Ex. 1). Using the same notation as there, $\mathbf{G}$ is then a **simple** graph whose connected components are simple star graphs S_2, see Figure 1. Also notice that $V(\mathbf{G}) = A$, whereas $\varrho^{\mathbf{A}}$ is the complement of the adjacency relation in $\mathbf{G}$. Theoretically, with respect to Proposition 2.7, in order to determine whether $\mathbf{A}$ is polymorphism-homogeneous one would need to consider all of its finite powers and check whether they are all homomorphism-homogeneous. However, as soon as we encounter a problem, be it that it is for the second power already, we can instantly conclude that the observed structure does not possess our desired property. Such is the case for the considered graph $\mathbf{A}$! Taking its square $\mathbf{A}^2$, see

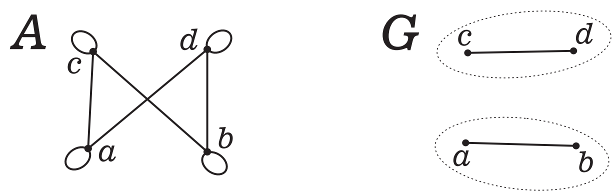

Figure 1: graphs **A** and **G**

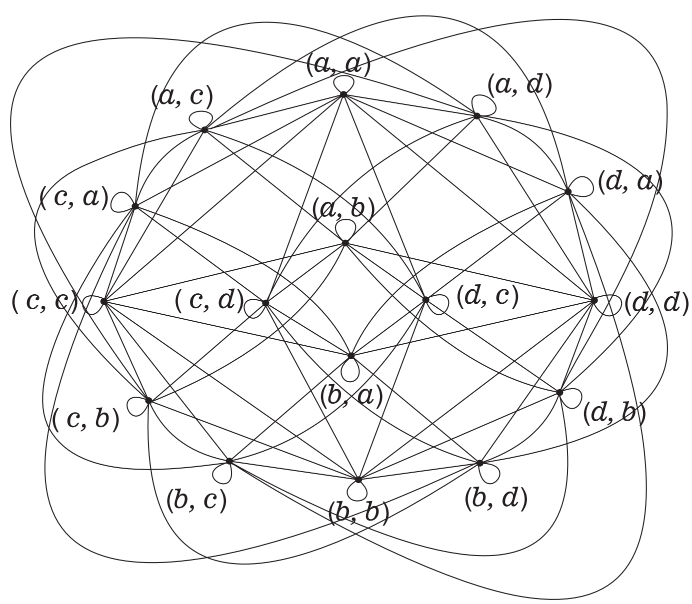

Figure 2: the square of graph **A**

Figure 2, we already spot a witness in the following form $(\mathbf{B_1}, \mathbf{B_2}, f, (b, a))$, where

$$B_1 := \{(c, c), (c, d), (d, c), (d, d)\}, B_2 := \{(c, a), (a, b), (d, a), (b, b)\},$$

and $f : (c, c) \mapsto (c, a); (c, d) \mapsto (a, b); (d, c) \mapsto (d, a); (d, d) \mapsto (b, b)$. Both

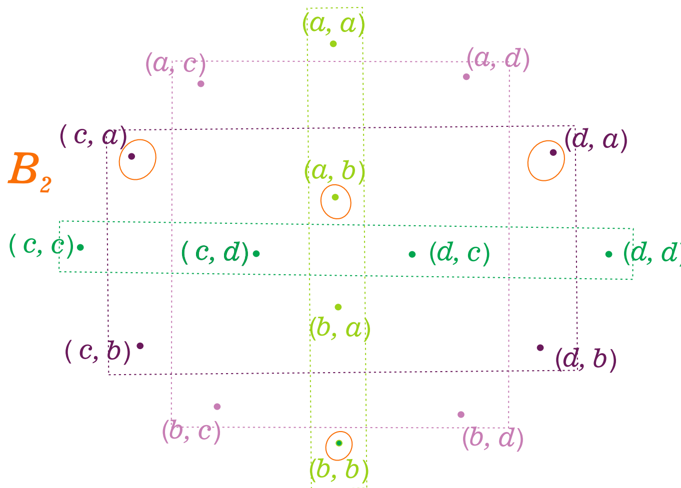

Figure 3: $\mathbf{A}^2$, with its maximal sets of independent vertices, each represented in a different colour

set B_1 and B_2 consist of independent vertices, see Figure 3. Following [16] (3.1.4. Prop. 1) we know that f had to be a bijection. Nevertheless, the vertices in B_2 have no common neighbour, unlike the ones in B_1, implying that f cannot be extended to $B_1 \cup \{(b, a)\}$. On balance, $(\mathbf{B_1}, \mathbf{B_2}, f, (b, a))$ is indeed a witness, thus proving $\mathbf{A}^2$ to be non-homomorphism-homogeneous.

It is now worthwhile considering whether there were any indications in $\mathbf{A}$ itself of the future occurrence of a witness which could have spared us the construction of its second power. For the purpose of gaining some insight and providing an answer to that let (x, y) be a common neighbour of all the vertices in B_2, as shown in Figure 4. Having in mind the definition of adjacency relation in $\mathbf{A}^2$ which heavily relies upon the same relation in $\mathbf{A}$ and is componentwise, we turn our attention to the first and second projections. We come to realise that if such an x were to exist it would necessarily be a common neighbour of all the vertices in A. The second conclusion related to y being a common neighbour of both a and b can trivially be satisfied by placing either c or d in place of y. However, the problem with x is that it would clearly have to belong to one of the two

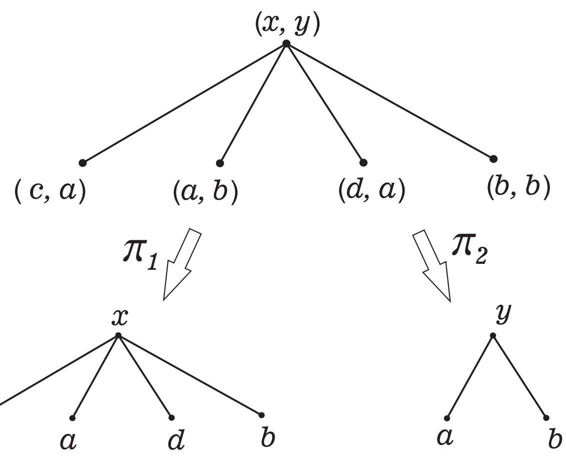

Figure 4: a hypothetical common neighbour of the vertices in B_2

bipartitions ($\{a, b\}$ or $\{c, d\}$), meaning that it could not be connected to the other member of the same. Therefore this would lead to a contradiction with the initial assumption!

Example 3.3 Consider the poset $\mathbf{A} = (A, \leqslant)$ shown in Figure 5. Notice

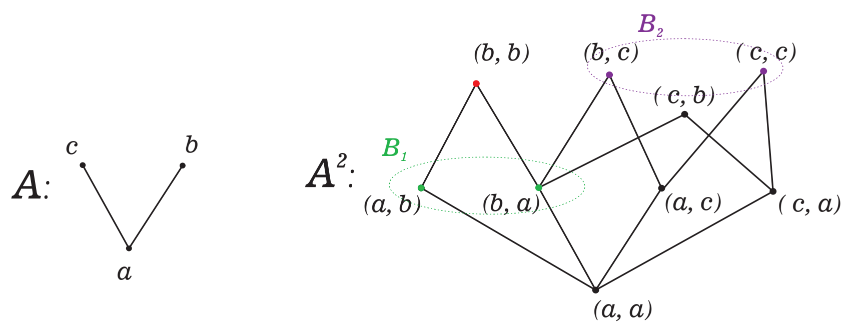

Figure 5: Poset **A** with its non-HH square

how its square $\mathbf{A}^2$ contains a witness, thus undeniably implying that A is not polymorphism-homogeneous. To paint the picture, $(\mathbf{B_1}, \mathbf{B_2}, f, (b, b))$ is one such, where $B_1 := \{(a, b), (b, a)\}$, $B_2 := \{(b, b), (c, c)\}$, and $f : (a, b) \mapsto (b, b); (b, a) \mapsto (b, c)$. Had there existed a common upper neighbour (x, y) of (b, b) and (b, c) then we would not have had a witness. For that to happen x would have to be an upper neighbour of b and b, whereas y would have to be the upper neighbour of b and c, see Figure 6. However, the latter of the

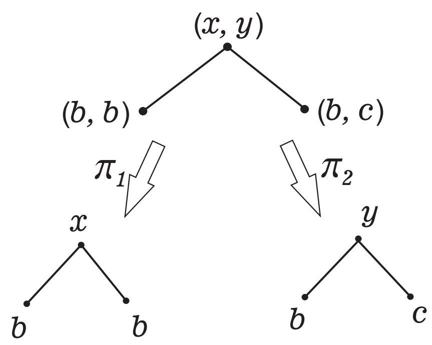

Figure 6: a hypothetical common neighbour of (b, b) and (b, c)

two cannot be satisfied. Finally, we may conclude that in order to avoid the occurrence of a witness then whenever there exists an infimum for any two elements of a poset, a supremum must exist for them just as well!

4 Metric Spaces

Definition 4.1 ([22]) Suppose M is a set and $d_{\mathcal{M}}$ is a real function defined on the Cartesian product $M \times M$. Then $d_{\mathcal{M}}$ is called a *metric* on M if, and only if, for each $a, b, c \in M$

(M1) $d_{\mathcal{M}}(a, b) \geqslant 0$ with equality if, and only if, $a = b$;

(M2) $d_{\mathcal{M}}(a, b) = d_{\mathcal{M}}(b, a)$; (*symmetric property*)

(M3) $d_{\mathcal{M}}(a, b) \leqslant d_{\mathcal{M}}(a, c) + d_{\mathcal{M}}(c, b)$.(*triangle inequality*)

We call the set M endowed with this metric a *metric space* and denote it by $\mathcal{M} := (M, d_{\mathcal{M}})$. For each $a, b \in M$, we call the number $d_{\mathcal{M}}(a, b)$ the *distance* between a and b with respect to the metric $d_{\mathcal{M}}$. Additionally, $\mathcal{M}$ is called an *S-metric space* whenever $im(d_{\mathcal{M}}) \subseteq S$.

Definition 4.2 A metric space $\mathcal{M}$ is called a *normalised metric space* if the smallest nonzero distance in $\mathcal{M}$ is 1, in case it exists.

Definition 4.3 Let $\mathcal{M}_1 := (M_1, d_{\mathcal{M}_1})$ and $\mathcal{M}_2 := (M_2, d_{\mathcal{M}_2})$ be two metric spaces. A mapping $f : M_1 \to M_2$ is said to be *k-Lipschitz* if for all $x, y \in M_1$

$$d_{\mathcal{M}_2}(f(x), f(y)) \leqslant k \cdot d_{\mathcal{M}_1}(x, y).$$

A *1-Lipschitz* mapping is also called *non-expansive.*

With a view to establishing what the most adequate homomorphisms of metric spaces are we introduce the following relational signature $L := (R, ar)$, all of whose relational symbols are of arity 2 and where

$$R := \{\rho_r : r \text{ is a non-negative real number}\}.$$

As it turns out metric spaces can be seen as certain L-structures. This becomes ever more evident upon fixing a metric space $\mathcal{M} = (M, d_{\mathcal{M}})$ and interpreting the symbol ρ_r, for each non-negative real number r, as below:

$$\forall x, y \in M \;\; (x, y) \in \rho_r^{\mathcal{M}} \text{ if, and only if, } d_{\mathcal{M}}(x, y) \leqslant r.$$

It now becomes clear why the metric introduced for some positive power of a metric space $\mathcal{M} = (M, d_{\mathcal{M}})$ has to be the *maximum metric*. In other words, that for a fixed $k \in \mathbb{N}_+$ and any $\bar{x}, \bar{y} \in M^k$

$$d_{\mathcal{M}^k}(\bar{x}, \bar{y}) := \max\{d_{\mathcal{M}}(x_i, y_i) : i \in \{1, 2, \ldots, k\}\}.$$

To be ever more precise, as by definition

$$\rho_r^{\mathcal{M}^k} = \{((a_{1,i})_{i \leqslant k}, (a_{2,i})_{i \leqslant k}) : \forall i \leqslant k (a_{1,i}, a_{2,i}) \in \rho_r^{\mathcal{M}}\}$$

then

$$\begin{aligned} d_{\mathcal{M}^k}(\bar{x}, \bar{y}) \leqslant r &\iff (\bar{x}, \bar{y}) \in \rho_r^{\mathcal{M}^k} \\ &\iff \forall i \leqslant k \;\; (x_i, y_i) \in \rho_r^{\mathcal{M}} \\ &\iff \forall i \leqslant k \;\; d_{\mathcal{M}}(x_i, y_i) \leqslant r. \end{aligned}$$

Lemma 4.4 *Homomorphisms of metric spaces understood as L-structures are precisely the non-expansive mappings.*

Proof. Having presented metric spaces in the language of model theory, i.e. as relational structures, as above, we now have that for any two metric spaces $\mathcal{M}_1 := (M_1, d_{\mathcal{M}_1})$ and $\mathcal{M}_2 := (M_2, d_{\mathcal{M}_2})$ a mapping $h : M_1 \to M_2$ is a homomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$ if, and only if, for any $r \in [0, \infty)$ and any $x_1, x_2 \in M_1$ if

$$(x_1, x_2) \in \rho_r^{\mathcal{M}_1} \text{ then } (h(x_1), h(x_2)) \in \rho_r^{\mathcal{M}_2},$$

or better yet if

$$d_{\mathcal{M}_1}(x_1, x_2) \leqslant r \text{ then } d_{\mathcal{M}_2}(h(x_1), h(x_2)) \leqslant r.$$

What is more, for any $x_1, x_2 \in M_1$ we have that

$$d_{\mathcal{M}_2}(h(x_1), h(x_2)) \leqslant d_{\mathcal{M}_1}(x_1, x_2),$$

hence h is non-expansive.

On the other hand, notice how every non-expansive mapping is trivially a homomorphism between any two metric spaces perceived as L-structures, due to the transitivity of the $\leqslant$ order on real numbers. □

Example 4.1 Any graph $\mathbf{G}$ can be perceived as a metric space $(V(\mathbf{G}), d)$ when defining its metric in the following manner. For any $x, y \in V(\mathbf{G})$ let

$$d(x,y) := \begin{cases} 0, \text{ when } x = y; \\ 1, \text{ when } x \neq y \text{ but } (x,y) \in E(\mathbf{G}); \\ 2, \text{ when } (x,y) \notin E(\mathbf{G}). \end{cases}$$

It follows, rather straightforwardly, that non-expansive mappings of such metric spaces correspond to the usual graph homomorphisms.

We know for a fact that deciding whether a finite metric space with rational distances is homomorphism-homogeneous is a **coNP**-complete problem [13], which actually results from [21]. On the other hand, one infinite case offers a rather satisfactory result stated below:

Theorem 4.5 (Dolinka 2012 [7]) *The rational Urysohn space is homomorphism-homogeneous.*

This could have been expected considering that the countable *random graph* R (the *'Rado graph'* [17]) turned out to be homomorphism-homogeneous, in the first place (see [4]), whereas the two have a number of similarities in common, see [2]. Recall that aside from being a countable graph R is characterised by the property that for any two disjoint finite sets of vertices U and V, there is a vertex z joined to all the vertices in U and to none in V (for further insight, see [3]).

Remark. Upon realising that graphs may, in fact, be represented as metric spaces (see Example 4.1) and again keeping Theorem 1.1 in mind, we initially wanted to avoid the class of such metric spaces which contains the graphs. This was the motivation behind the introduction of the *$\star$-property.*

Definition 4.6 A metric space $\mathcal{M}$ for which there exist no such $a, b > 0$ in $\mathrm{im}(d_{\mathcal{M}})$ satisfying the condition that $a < b \leqslant 2 \cdot a$ is said to have the *$\star$-property*, and referred to as a *$\star$-metric space.*

Remark. Notice how in any $\star$-metric space $\mathcal{M}$ for any two positive numbers $a, b \in \text{im}(d_{\mathcal{M}})$ it trivially holds that:

$$\text{if } \; b \leqslant 2a \; \text{ then } \; b \leqslant a.$$

It is now fairly evident that the class of metric spaces that do **not** have the $\star$-property "includes" graphs. Fixing any $a, b > 0$ that $a < b \leqslant 2 \cdot a$, any graph may be interpreted as a metric space $\mathcal{M}$ with $\text{im}(d_{\mathcal{M}}) = \{0, a, b\}$ similarly as in Example 4.1, but having a in place of 1 and b in place of 2. Hence,

the problem of deciding whether a metric space **without** the $\star$-property is homomorphism-homogeneous or not is a **coNP**-complete problem,

as a direct consequence of Theorem 1.1.

However, considering the case of $\star$-metric spaces is not nearly as hopeless as that. Not only are such countable metric spaces homomorphism-homogeneous, but they are polymorphism-homogenous, as well, as we will show below.

4.1 Metric spaces with the $\star$-property

Lemma 4.7 *Let $\mathcal{M}$ be a $\star$-metric space. Then $\rho_r^{\mathcal{M}}$ is an equivalence relation on M, for any $r \in \text{im}(d_{\mathcal{M}})$.*

Proof. Let $x, y, z \in M$. Then trivially $d_{\mathcal{M}}(x, x) = 0 \leqslant r$ implying that $(x, x) \in \rho_r^{\mathcal{M}}$, which proves the reflexivity. Further on, supposing that $(x, y) \in \rho_r^{\mathcal{M}}$ we have that $d_{\mathcal{M}}(x, y) \leqslant r$ which is the same as $d_{\mathcal{M}}(y, x) \leqslant r$ and consequently equivalent to $(y, x) \in \rho_r^{\mathcal{M}}$, thus proving this relation to be symmetric. Finally, in order to prove the transitivity assume at first that $(x, y) \in \rho_r^{\mathcal{M}}$ and $(y, z) \in \rho_r^{\mathcal{M}}$. Due to the triangle inequality, the following holds: $d_{\mathcal{M}}(x, z) \leqslant d_{\mathcal{M}}(x, y) + d_{\mathcal{M}}(y, z)$. Additionally, we have that $d_{\mathcal{M}}(x, y) \leqslant r$ and $d_{\mathcal{M}}(y, z) \leqslant r$ leading to $d(x, z) \leqslant 2 \cdot r$. Now, as $\mathcal{M}$ possesses the $\star$-property we have $d(x, z) \leqslant r$, and therefore $(x, z) \in \rho_r^{\mathcal{M}}$. $\square$

Remarks. Let $\mathcal{M}$ be a $\star$-metric space.

(i) When $r = \min(\text{im}(d_{\mathcal{M}}) \backslash \{0\})$ exists and $(x, y) \in \rho_r^{\mathcal{M}}$ then $d(x, y) = r$ (for $x \neq y$).

(ii) For any $p, q \in \text{im}(d_{\mathcal{M}})$, such that $p \leqslant q$, assuming that $(u, v) \in \rho_p^{\mathcal{M}}$ implies $(u, v) \in \rho_q^{\mathcal{M}}$ since clearly $d_{\mathcal{M}}(u, v) \leqslant p \leqslant q$.

Theorem 4.8 *All finite and countably infinite $\star$-metric spaces are homomorphism-homogeneous.*

Proof. Consider an arbitrary $\star$-metric space $\mathcal{M} = (M, d_{\mathcal{M}})$ with $|M| \leqslant \aleph_0$. With a view to proving that $\mathcal{M}$ possesses the one-point extension property, take any local homomorphism $f : T \to M$ with the domain $T := \{x_1, x_2, \ldots, x_k\}$ where $x_i \neq x_j$ for $i \neq j$. Further, take any $z \in M \backslash T$ and let m be the minimal distance from z to any point in T. Additionally, let

$$j_z := \min\{i \in \{1, 2, \ldots, k\} \mid d_{\mathcal{M}}(z, x_i) = m\}.$$

We may now define an extension $g : T \cup \{z\} \to M$ of f, so that $g(z) := f(x_{j_z})$, whereas $g(x) := f(x)$ for all $x \in T$. What suffices to show is that g is in fact a homomorphism. Therefore, we take any $x, y \in T \cup \{z\}$ and consider the following two cases:

(i) If $x, y \in T$ then

$$d_{\mathcal{M}}(g(x), g(y)) = d_{\mathcal{M}}(f(x), f(y)) \leqslant d_{\mathcal{M}}(x, y),$$

having taken into account that g acts the same way on T as f, which is already a homomorphism.

(ii) If $x \in T$, but $y = z$ then

$$d_{\mathcal{M}}(g(x), g(y)) = d_{\mathcal{M}}(g(x), g(z)) = d_{\mathcal{M}}(f(x), f(x_{j_z})) \leqslant d_{\mathcal{M}}(x, x_{j_z}).$$

All that remains it to show that $d_{\mathcal{M}}(x, x_{j_z}) \leqslant d_{\mathcal{M}}(x, y)$. Indeed, combining the $\star$-property with

$$d_{\mathcal{M}}(x, x_{j_z}) \leqslant d_{\mathcal{M}}(x, y) + d_{\mathcal{M}}(y, x_{j_z}) \leqslant d_{\mathcal{M}}(x, y) + d_{\mathcal{M}}(y, x) = 2d_{\mathcal{M}}(x, y),$$

we obtain the desired inequality.

All this leads to the conclusion that g is an extension of f and that $\mathcal{M}$ thus has the one-point extension property. The homomorphism-homogeneity of it comes as a consequence of Proposition 2.9. □

Theorem 4.9 *All finite and countably infinite $\star$-metric spaces are polymorphism-homogeneous.*

Proof. Let $\mathcal{M} = (M, d)$ be a finite (or at most a countably infinite) $\star$-metric space. Notice how, for any fixed $k \in \mathbb{N}_+$, the ordered pair $(M^k, d_{\mathcal{M}^k}) =: \mathcal{M}^k$ is again a $\star$-metric space. This is due to $\mathrm{im}(d_{\mathcal{M}^k}) = \mathrm{im}(d_{\mathcal{M}})$. As such, according to Theorem 4.8 it is homomorphism-homogeneous! Thus, applying Proposition 2.7 we have that $\mathcal{M}$ is k-polymorphism-homogeneous. Since k was an arbitrary positive integer it immediately follows that $\mathcal{M}$ is a polymorphism-homogeneous metric space, by the definition of polymorphism-homogeneity. $\square$

4.2 Metric spaces without the $\star$-property

Turning our attention to the class of metric spaces which do not possess the $\star$-property, we recall the statement of Theorem 1.1. The insight we gain, is that the decision problem with respect to homomorphism-homogeneity of such metric spaces is substantially difficult. Whilst the full classification is thus admittedly a bit far-fetched, there is seemingly no solid argument against the full classification of such countable metric spaces with respect to polymorphism-homogeneity. If that were to be shown formally, then together with the results from the previous subsection the observed problem would be solved, irrespective of the $\star$-property.

4.2.1 $n\star$-metric spaces with connected skeletons

In order to shed some further light on metric spaces without the $\star$-property, we will begin by investigating such **finite** metric spaces. What is more, without loss of generality, we shall only consider the **normalised** ones amongst those and refer to them as "*$n\star$-metric spaces*". The reasons behind this simplification lie in the observation that any finite metric space $\mathcal{M} = (M, d_{\mathcal{M}})$ has at most $\binom{|M|}{2}$ different distances in it, meaning that $\mathrm{im}(d_{\mathcal{M}})$ is finite. This allows us to freely scale the distances in $\mathcal{M}$. Consequently, $\mathcal{M}$ can be seen as a metric space where $\mathrm{im}(d_{\mathcal{M}}) = \{0, 1, d_2, \ldots, d_k\}$, for some $k \in \mathbb{N}_+ \backslash \{1\}$ and $1 < d_2 < \cdots < d_k$.

Standing assumption: From this moment on all the metric spaces we work with are finite.

Definition 4.10 To each such $\mathcal{M}$, as above described, corresponds a unique graph $\mathbf{G}_{\mathcal{M}}$ such that $V(\mathbf{G}_{\mathcal{M}}) = M$, whereas:

$$(x, y) \in E(\mathbf{G}_{\mathcal{M}}) \text{ if and only if } d_{\mathcal{M}}(x, y) \in \{0, 1\}, \text{ for any } x, y \in M.$$

We shall refer to $\mathbf{G}_{\mathcal{M}}$ as the *skeleton* of $\mathcal{M}$. The usual graph metric on it, introduced in Section 2.3, will here be denoted by $\delta_{\mathcal{M}}$.

An $n\star$-metric space with a **connected** skeleton will be called a "*c-metric space*."

Remark. The dual nature of the elements of M will not be explicitly discussed further on. It shall always be perfectly clear from the context when they are referred to as the points of $\mathcal{M}$, and when as the vertices of $\mathbf{G}_{\mathcal{M}}$.

Lemma 4.11 *Every local homomorphism of an $n\star$-metric space $\mathcal{M}$ is at the same time a local homomorphism of its skeleton.*

Proof. Let f be a local homomorphism of $\mathcal{M}$ and take any $u, v \in \mathrm{dom}(f)$, such that $(u, v) \in E(\mathbf{G}_{\mathcal{M}})$. Further, we distinguish the following two cases:

1) When (u, v) is a loop in $\mathbf{G}_{\mathcal{M}}$, then $u = v$, and so $d_{\mathcal{M}}(u, v) = 0$. Moreover,

$$d_{\mathcal{M}}(f(u), f(v)) \leqslant d_{\mathcal{M}}(u, v) = 0,$$

from the non-expansiveness of f, leading to $d_{\mathcal{M}}(f(u), f(v)) = 0$, which is equivalent to $f(u) = f(v)$. It only remains to notice that $f[(u, v)] = (f(u), f(v))$ is now a loop.

2) When (u, v) is a proper edge in $\mathbf{G}_{\mathcal{M}}$, then $d_{\mathcal{M}}(u, v) = 1$. Again, due to the non-expansiveness of f, we have that

$$d_{\mathcal{M}}(f(u), f(v)) \leqslant d_{\mathcal{M}}(u, v) = 1.$$

By the definition of the skeleton of $\mathcal{M}$, $(f(u), f(v)) = f[(u, v)] \in E(\mathbf{G}_{\mathcal{M}})$.

What we have thus shown is that f is also a local homomorphism of $\mathbf{G}_{\mathcal{M}}$. □

The converse of Lemma 4.11 does not hold in general, as can be seen from the Example 4.2 below.

Example 4.2 Consider the $n\star$-metric space $\mathcal{M}$ shown in Figure 7 and define a local homomorphism h of $\mathbf{G}_{\mathcal{M}}$ as such that it maps $u \mapsto u$ and $w_2 \mapsto v$. Realising that

$$d_{\mathcal{M}}(u, w_2) = b < a = d_{\mathcal{M}}(u, v) = d_{\mathcal{M}}(h(u), h(w_2)),$$

we immediately have that h is not 1-Lipschitz. Consequently, it is not a local homomorphism of $\mathcal{M}$, at all!

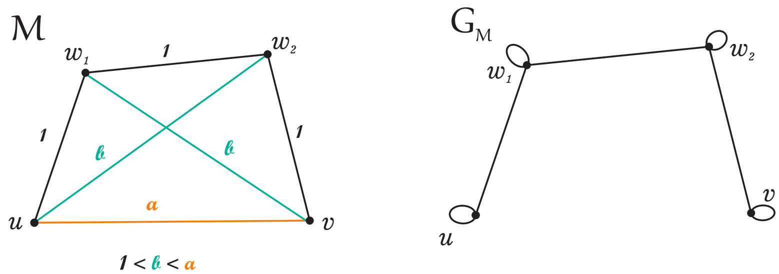

Figure 7: A metric space $\mathcal{M}$ with its skeleton $G_{\mathcal{M}}$

Another consequence which may be derived from Lemma 4.11 is that, unfortunately, we **cannot** conclude that an $n\star$-metric space $\mathcal{M}$ is homomorphism-homogeneous in case its skeleton is such. What Lemma 4.11 implies is that every local homomorphism f of $\mathcal{M}$ may be extended to an endomorphism g of $\mathbf{G}_{\mathcal{M}}$, when $\mathbf{G}_{\mathcal{M}}$ is homomorphism-homogenous. However, we have no guaranty that $g \in \mathrm{End}(\mathcal{M})$, see Example 4.3.

Example 4.3 Consider the $n\star$-metric space $\mathcal{M}$ provided in Figure 8. De-

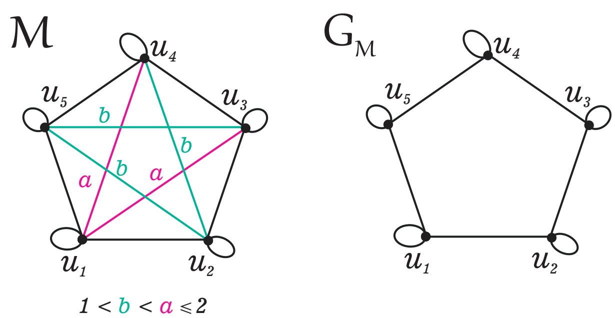

Figure 8: A non-HH metric space $\mathcal{M}$ with a HH skeleton

fine a local homomorphism f of $\mathcal{M}$ so that it maps u_1, u_2, u_3, u_4 onto u_5, u_4, u_3, u_2, respectively. If there were a $g \in \mathrm{End}(\mathcal{M})$ extending f, then $g(u_5)$ would have to be connected to both $g(u_1) = u_5$ and $g(u_4) = u_2$, leaving us with no other option but $g(u_5) = u_1$. However,

$$d_{\mathcal{M}}(g(u_3), g(u_5)) = d_{\mathcal{M}}(u_3, u_1) = a > b = d_{\mathcal{M}}(u_3, u_5),$$

which collides with g's non-expansiveness. On the other hand, notice how $g \in \mathrm{End}(\mathbf{G}_{\mathcal{M}})$. The homomorphism-homogeneity of $\mathbf{G}_{\mathcal{M}}$ comes from [16].

Definition 4.12 Let $\mathcal{M}$ be an $n\star$-metric space. An $s \in \operatorname{im}(d_{\mathcal{M}})$ is called a *k-distance* in $\mathcal{M}$ if there exist such $x, y \in M$ that $d_{\mathcal{M}}(x, y) = s$ and $\delta_{\mathcal{M}}(x, y) = k$.

Definition 4.13 The set of points of an $n\star$-metric space $\mathcal{M}$ is called a (k)-*set* in $\mathcal{M}$ if all the graph distances in $\mathbf{G}_{\mathcal{M}}$ between any two elements are at most k.

Remark. A subgraph of $\mathbf{G}_{\mathcal{M}}$ induced by a (1)-set in $\mathcal{M}$ corresponds to none other than a complete subgraph in $\mathbf{G}_{\mathcal{M}}$.

Lemma 4.14 *Let $\mathcal{M} := (M, d_{\mathcal{M}})$ be an $n\star$-metric space. Then, every non-expansive mapping of $\mathcal{M}$ to $\mathcal{M}$ is also a non-expansive mapping of $(M, \delta_{\mathcal{M}})$ to $(M, \delta_{\mathcal{M}})$.*

Proof. Let f be a non-expansive mapping of $\mathcal{M}$ to $\mathcal{M}$. Suppose the opposite, that there exist such $u, v \in M$ for which $\delta_{\mathcal{M}}(f(u), f(v)) > \delta_{\mathcal{M}}(u, v) =: k$. In case $k > 1$, there exists a path $uw_1w_2 \ldots w_{k-1}v$ in $\mathbf{G}_{\mathcal{M}}$ of length k. However, the walk $f(u)f(w_1)f(w_2)\ldots f(w_{k-1})f(v)$ is a path in $\mathbf{G}_{\mathcal{M}}$ of length at most k. Therefore, $\delta_{\mathcal{M}}(f(u), f(v)) \leqslant k$ which is a clear contradiction. On the other hand, if $k = 1$ then $d_{\mathcal{M}}(u, v) = 1$ and owing to the non-expansiveness of f we have $d_{\mathcal{M}}(f(u), f(v)) \leqslant 1$ implying nothing less then $\delta_{\mathcal{M}}(f(u), f(v)) \leqslant 1 = d_{\mathcal{M}}(u, v)$ which once again collides with our initial assumption. $\square$

Proposition 4.15 *Any homomorphism-homogeneous $n\star$-metric space $\mathcal{M}$ satisfies the following condition:*

For all $u, v, s, t \in M$ if $\delta_{\mathcal{M}}(u, v) < \delta_{\mathcal{M}}(s, t) < \infty$ then $d_{\mathcal{M}}(u, v) < d_{\mathcal{M}}(s, t)$.

Proof. At first, we fix some arbitrary vertices $u, v, s, t \in M$ for which

$$a := \delta_{\mathcal{M}}(u, v) < \delta_{\mathcal{M}}(s, t) =: b.$$

- If $a = 0$ then $u = v$ and $\delta_{\mathcal{M}}(s, t) > 0$, so $s \neq t$. Consequently, $d_{\mathcal{M}}(u, v) = 0 < d_{\mathcal{M}}(s, t)$.
- If $a = 1$ then by the definition of $\mathbf{G}_{\mathcal{M}}$ the distance $d_{\mathcal{M}}(u, v) = 1$ as well, whereas, due to $b > a = 1$, $d_{\mathcal{M}}(s, t) > 1 = d_{\mathcal{M}}(u, v)$.

- If $a \geqslant 2$, then there exist such $w_1, w_2, \ldots, w_{a-1} \in M$ that $uw_1w_2 \ldots w_{a-1}v$ is a path of length a connecting u and v. Obviously, $\delta_{\mathcal{M}}(u, w_{a-1}) = a-1$. Assuming the opposite that $d_{\mathcal{M}}(u,v) \geqslant d_{\mathcal{M}}(s,t)$, we define a local homomorphism f of $\mathcal{M}$ mapping u onto s and v onto t. Owing to the homomorphism-homogeneity of $\mathcal{M}$ there exists an endomorphism g of $\mathcal{M}$ which expands f. Moreover,

$$\delta_{\mathcal{M}}(s, g(w_{a-1})) = \delta_{\mathcal{M}}(g(u), g(w_{a-1})) \leqslant \delta_{\mathcal{M}}(u, w_{a-1}) = a - 1,$$

whereas

$$\delta_{\mathcal{M}}(g(w_{a-1}), t) = \delta_{\mathcal{M}}(g(w_{a-1}), g(v)) \leqslant \delta_{\mathcal{M}}(w_{a-1}, v) = 1.$$

This, however, leads to the conclusion that

$$b := \delta_{\mathcal{M}}(s,t) \leqslant \delta_{\mathcal{M}}(s, g(w_{a-1})) + \delta_{\mathcal{M}}(g(w_{a-1}), t) \leqslant (a-1) + 1 = a$$

which is a clear contradiction.

□

Corollary 4.16 *In any homomorphism-homogeneous $n\star$-metric space $\mathcal{M}$:*

An i-distance $<$ j-distance if and only if $i < j$,

for $i \neq j$, where $i, j \in \mathbb{N}$.

Proof. The opposite implication is equivalent to the statement of Proposition 4.15, whereas the direct implication is proven simply by the means of contraposition. □

The above Corollary 4.16 helps us create an overall better understanding of the non-expansive mappings. It basically says that an i-distance can be "homomorphically mapped" onto a j-distance, whenever $j < i$. This is why grouping distances from S into separate disjoint classes of k-distances, for $k \in \mathbb{N}_+$, actually makes sense. However, this does not mean that all the i-distances in a homomorphism-homogeneous $n\star$-metric space have to be equal. In Theorem 4.47, for example, there is no limit as to how many different 3-distances can appear.

Definition 4.17 Let $\mathcal{M}$ be a c-metric space, with $\operatorname{diam}(\mathbf{G}_{\mathcal{M}}) \geqslant 3$. For any path $u_1u_2 \ldots u_{i+1}$ of length $i \geqslant 3$ in $\mathbf{G}_{\mathcal{M}}$, as in Figure 9, we say that $d_{\mathcal{M}}(u_1, u_i)$ and $d_{\mathcal{M}}(u_2, u_{i+1})$ are $(i-1)$*-distances over* (u_1, u_{i+1})

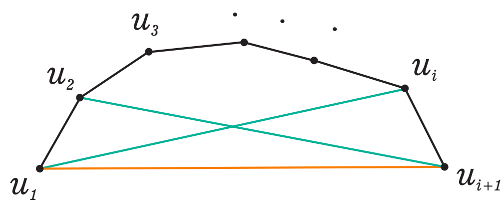

Figure 9: $(i-1)$-distances over an i-distance

Remark. Keeping the same notation as in the above definition both $d_{\mathcal{M}}(u_1, u_i)$ and $d_{\mathcal{M}}(u_2, u_{i+1})$ are indeed $(i-1)$-distances, to begin with. Evidently they are at most $(i-1)$-distances, due to the paths $u_1u_2 \ldots u_i$ and $u_2u_3 \ldots u_{i+1}$, respectively. Additionally, had either one of them been of length $j < i-1$ there would have thus existed a path of length $j+1 < i$ between u_1 and u_{i+1}, which is impossible as $d_{\mathcal{M}}(u_1, u_{i+1})$ is strictly an i-distance!

Proposition 4.18 *Let $\mathcal{M}$ be a homomorphism-homogeneous c-metric space and $u, v \in M$ such that $\delta_{\mathcal{M}}(u, v) = i$, for some $i \geqslant 3$. Let b be the shortest $(i-1)$-distance over (u, v). Then there exists a path of length i connecting u and v such that both of the $(i-1)$-distances over (u, v) determined by it are precisely b.*

Proof. Take an arbitrary path $uw_1w_2 \ldots w_{i-1}v$ of length i in $\mathbf{G}_{\mathcal{M}}$, such that at least one of the $(i-1)$-distances is b, see Figure 10. Without loss od generality assume that $d_{\mathcal{M}}(u, w_{i-1}) = b$ and $d_{\mathcal{M}}(v, w_1) =: c \geqslant b$. Then,

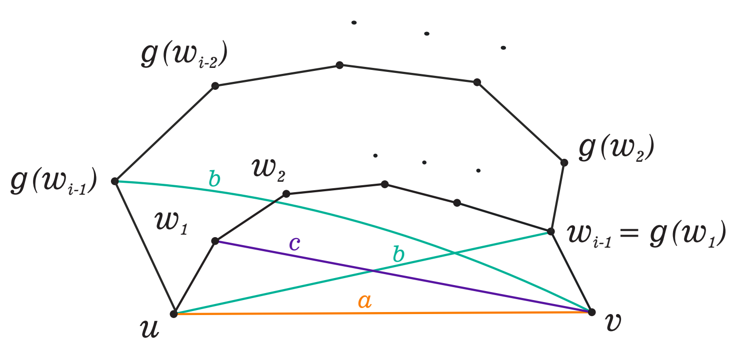

Figure 10: two $(i-1)$-distances b over (u, v)

the mapping $f : u \mapsto v; \quad v \mapsto u; \quad w_1 \mapsto w_{i-1}$, is a well-defined local homomorphism. Since $\mathcal{M}$ is homomorphism-homogeneous there exists an endomorphic extension g of f. Consequently, $g(w_{i-1})$ has got to be at distance 1 from u since

$$d_{\mathcal{M}}(g(w_{i-1}), u) = d_{\mathcal{M}}(g(w_{i-1}), g(v)) \leqslant d_{\mathcal{M}}(w_{i-1}, v) = 1;$$

and $g(w_{i-1}) \neq u$, because otherwise

$$d_{\mathcal{M}}(u, v) = d_{\mathcal{M}}(g(w_{i-1}), g(u)) \leqslant d_{\mathcal{M}}(w_{i-1}, u) = b,$$

which is a clear contradiction with Proposition 4.15. Furthermore, since $d_{\mathcal{M}}(u, w_{i-1}) = b$ is an $(i-1)$-distance in $\mathcal{M}$ then $d_{\mathcal{M}}(g(u), g(w_{i-1})) = d_{\mathcal{M}}(v, g(w_{i-1}))$ is at most an $(i-1)$-distance in $\mathcal{M}$. In case it were a j-distance in $\mathcal{M}$, for $j < i-1$, there would have to exist some path in $\mathbf{G}_{\mathcal{M}}$ of length j between $g(w_{i-1})$ and v. Yet attaching u to the beginning of it would yield a path of length $1+j < 1+(i-1) = i$ connecting u and v, which is a clear contradiction with $\delta_{\mathcal{M}}(u, v) = i$. Consequently, $\delta_{\mathcal{M}}(v, g(w_{i-1}))$ is an $(i-1)$-distance over (u, v) in $\mathcal{M}$ and so it is $\geqslant b$. Additionally:

$$d_{\mathcal{M}}(v, g(w_{i-1})) = d_{\mathcal{M}}(g(u), g(w_{i-1}))) \leqslant d_{\mathcal{M}}(u, w_{i-1}) = b,$$

resulting in $d_{\mathcal{M}}(v, g(w_{i-1})) = b$. Finally, notice how

$$g(v)g(w_{i-1})g(w_{i-2}) \dots g(w_1)g(u)$$

is then not only a walk of length i, but also a path in $\mathbf{G}_{\mathcal{M}}$ connecting $g(v) = u$ and $g(u) = v$, due to $\delta_{\mathcal{M}}(u, v) = i$, see Figure 10 again.

□

Proposition 4.19 *Let $u_1, u_{i+1}, v_1, v_{i+1} \in M$ be such that $\delta_{\mathcal{M}}(u_1, u_{i+1}) = \delta_{\mathcal{M}}(v_1, v_{i+1}) = i$, and define $a_1 := d_{\mathcal{M}}(u_1, u_{i+1})$, $a_2 := d_{\mathcal{M}}(v_1, v_{i+1})$. Let b_1 and b_2 be the minimal $(i-1)$-distances over (u_1, u_{i+1}) and (v_1, v_{i+1}), respectively.*

If $a_1 \geqslant a_2$ then $b_1 \geqslant b_2$.

Proof. Let $a_1 \geqslant a_2$. Owing to Lemma 4.18, there now exists some path $u_1 u_2 \dots u_{i+1}$ of length i within $\mathbf{G}_{\mathcal{M}}$, such that $d_{\mathcal{M}}(u_1, u_i) = d_{\mathcal{M}}(u_2, u_{i+1}) = b_1$, as shown in Figure 11. Evidently, the mapping $f : u_1 \mapsto v_1; u_{i+1} \mapsto v_{i+1}$ is a local homomorphism of $\mathcal{M}$! Since $\mathcal{M}$ is homomorphism-homogenous there exists a homomorphic extension g of f. Moreover, due to u_2 being connected to u_1 in $\mathbf{G}_{\mathcal{M}}$, $g(u_2)$ is then at a distance 1 from $g(u_1) = f(u_1) = v_1$. It was easy to see that $g(u_2) \neq v_1$, because otherwise

$$d_{\mathcal{M}}(g(u_2), g(u_{i+1})) = d_{\mathcal{M}}(v_1, v_{i+1}) = a_2 > b_1 = d_{\mathcal{M}}(u_2, u_{i+1}),$$

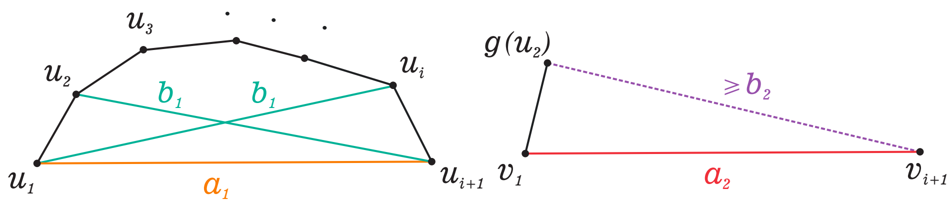

Figure 11: i-distances a_1 and a_2 in $\mathcal{M}$

coming into direct collision with the non-expansiveness of g. Additionally, as $d_{\mathcal{M}}(u_2, u_{i+1})$ is an $(i-1)$-distance in $\mathcal{M}$ and $g(u_{i+1}) = f(u_{i+1}) = v_{i+1}$ then $d_{\mathcal{M}}(g(u_2), g(u_{i+1}))$ is an $(i-1)$-distance over (v_1, v_{i+1}). (The very non-expansiveness does not allow it to go beyond an $(i-1)$-distance. Further, if it were at most an $(i-2)$-distance in $\mathcal{M}$ then there would have to exist some path connecting $g(u_2)$ and v_{i+1} of length at most $i-2$. However, adding v_1 at the beginning what that would imply is that the newly acquired path is of length at most $i-1$ over (v_1, v_{i+1}) — the i-distance a_2, which is impossible. Finally,

$$b_2 \leqslant d_{\mathcal{M}}(g(u_2), g(u_{i+1})) \leqslant d_{\mathcal{M}}(u_2, u_{i+1}) = b_1,$$

as b_2 is the smallest $(i-1)$-distance over (v_1, v_{i+1}). □

Corollary 4.20 *Let $\mathcal{M}$ be a homomorphism-homogenous $n\star$-metric space and $d_{\mathcal{M}}(u_1, v_1) = d_{\mathcal{M}}(u_2, v_2)$ be an i-distance in $\mathcal{M}$ for some $u_1, u_2, v_1, v_2 \in M$ and $i \geqslant 3$. Then, the minimal $(i-1)$-distance b_1 over (u_1, v_1) in $\mathcal{M}$ is equal to the minimal $(i-1)$-distance b_2 over (u_2, v_2) in $\mathcal{M}$.*

Proof. Simply by applying Proposition 4.19 twice we get that $b_1 = b_2$. □

The previous corollary justifies the following definition:

Definition 4.21 Let a be an i-distance and b an $(i-1)$-distance. We call b *the minimal $(i-1)$-distance over a* if for some u, v of distance a in $\mathcal{M}$ (and, hence, for all u, v of distance a in $\mathcal{M}$) b is the minimal $(i-1)$-distance over (u, v).

Next, we wish to see what properties $\mathcal{M}$ passes on to its direct powers.

Proposition 4.22 *Let $\mathcal{M} = (M, d_{\mathcal{M}})$ be an $n\star$-metric space. If, for all $s, t, u, v \in M$, the following condition holds:*

$$\delta_{\mathcal{M}}(s,t) < \delta_{\mathcal{M}}(u,v) < \infty \Longrightarrow d_{\mathcal{M}}(s,t) < d_{\mathcal{M}}(u,v),$$

then for each positive integer n an analogous implication must hold for $\mathcal{M}^n$. In other words, for any $n \in \mathbb{N}_+$ and any $\bar{s}, \bar{t}, \bar{u}, \bar{v} \in M^n$, under the given assumption:

$$\delta_{\mathcal{M}^n}(\bar{s}, \bar{t}) < \delta_{\mathcal{M}^n}(\bar{u}, \bar{v}) < \infty \Longrightarrow d_{\mathcal{M}^n}(\bar{s}, \bar{t}) < d_{\mathcal{M}^n}(\bar{u}, \bar{v}).$$

Proof. Suppose the condition stated in this proposition is true for $\mathcal{M}$. Then, fix an $n \in \mathbb{N}_+$ and choose such $\bar{s}, \bar{t}, \bar{u}, \bar{v} \in M^n$ for which

$$a := \delta_{\mathcal{M}^n}(\bar{s}, \bar{t}) < \delta_{\mathcal{M}^n}(\bar{u}, \bar{v}) =: b.$$

Now, by the definition of maximum metric $d_{\mathcal{M}^n}$ there exist such i^* and j^* from $\{1, 2, \ldots, n\}$ for which $d_{\mathcal{M}^n}(\bar{s}, \bar{t}) = d_{\mathcal{M}}(s_{i^*}, t_{i^*})$ and $d_{\mathcal{M}^n}(\bar{u}, \bar{v}) = d_{\mathcal{M}}(u_{j^*}, v_{j^*})$. With that in mind, $d_{\mathcal{M}}(s_{i^*}, t_{i^*})$ is an a-distance in $\mathcal{M}$ whereas $d_{\mathcal{M}}(u_{j^*}, v_{j^*})$ is a b-distance in $\mathcal{M}$. Out of the very condition imposed on $\mathcal{M}$ at the start, and

$$\delta_{\mathcal{M}}(s_{i^*}, t_{i^*}) = a < b = \delta_{\mathcal{M}}(u_{j^*}, v_{j^*}),$$

we obtain the desired inequality:

$$d_{\mathcal{M}^n}(\bar{s}, \bar{t}) = d_{\mathcal{M}}(s_{i^*}, t_{i^*}) < d_{\mathcal{M}}(u_{j^*}, v_{j^*}) = d_{\mathcal{M}^n}(\bar{u}, \bar{v}).$$

□

Proposition 4.23 *Let $\mathcal{M} = (M, d_{\mathcal{M}})$ be an $n\star$-metric space and $n, i \in \mathbb{N}_+$. Then every i-distance in $\mathcal{M}$ is also an i-distance in $\mathcal{M}^n$.*

Proof. Let a be an i-distance in $\mathcal{M}$. Then there exist such $u, v \in M$ that $\delta_{\mathcal{M}}(u, v) = i$. This also implies the existence of a path $uw_1 \ldots w_{i-1}v$ in $\mathbf{G}_{\mathcal{M}}$ of length i. Further, we introduce $\bar{u} := (u, u, \ldots, u), \bar{v} := (v, v, \ldots, v)$ and $\bar{w}_j := (w_j, w_j, \ldots, w_j)$, for all $j \in \{1, \ldots, i-1\}$, all of which are points of $\mathcal{M}^n$. Evidently, $\bar{u}\bar{w}_1 \ldots \bar{w}_{i-1}\bar{v}$ is a path of length i in $\mathbf{G}_{\mathcal{M}^n}$ whereas

$$\delta_{\mathcal{M}^n}(\bar{u}, \bar{v}) = \delta_{\mathcal{M}}(u, v) = a.$$

Therefore, a is an i-distance in $\mathcal{M}$, as well. □

Proposition 4.24 *Let $\mathcal{M} = (M, d_{\mathcal{M}})$ be an $n\star$-metric space. If for some $i \in \mathbb{N}_+$, there exists only one i-distance in $\mathcal{M}$, then for any positive integer n, there is again precisely that one i-distance in $\mathcal{M}^n$.*

Proof. At first let a be the unique i-distance in $\mathcal{M}$ and fix an $n \in \mathbb{N}_+$. Proposition 4.23 implies that a is additionally an i-distance in $\mathcal{M}^n$. Assume, now, that some $\bar{x}, \bar{y} \in M^n$ are at an i-distance in $\mathcal{M}^n$. In other words that $\delta_{\mathcal{M}^n}(\bar{x}, \bar{y}) = i$. Denoting the set of all the indices $j \in \{1, 2, \ldots, n\}$ for which $d_{\mathcal{M}^n}(\bar{x}, \bar{y}) = d_{\mathcal{M}}(x_j, y_j)$ by J, we immediately have that $d_{\mathcal{M}}(x_j, y_j) = a$, for any $j \in J$. This was merely due to $\delta_{\mathcal{M}}(x_j, y_j) = i$, for any $j \in J$. □

A trivial remark. For any $n\star$-metric space $\mathcal{M}$ and any $i \in \mathbb{N}_+$, the union of all the maximal (i)-sets is always the whole M. In other words, if we were to take any point $x \in M$ it would belong to some maximal (i)-set. The reason for this is rather obvious as $\mathcal{M}$ is finite.

Proposition 4.25 *Let $\mathcal{M}$ be a c-metric space. In case the intersection of all of its maximal (i)-sets is a non-empty set, where $i \leqslant \operatorname{diam}(\mathbf{G}_{\mathcal{M}})$, then the same is true for the maximal (i)-sets of $\mathcal{M}^n$, for any $n \in \mathbb{N}_+$.*

Proof. Let $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_l$ be all the maximal (i)-sets in $\mathcal{M}$ for some fixed $i \leqslant \operatorname{diam}(\mathbf{G}_{\mathcal{M}})$. Then, there exists a $c \in \bigcap_{i=1}^{l} \mathcal{K}_i$, due to the very assumptions of this Proposition. Clearly, $\delta_{\mathcal{M}}(x, c) \leqslant i$ for any $x \in M$ as c belongs to all the maximal (i)-sets, so $d_{\mathcal{M}}(x, c)$ is at most an i-distance in $\mathcal{M}$. Now, fix a positive integer n and take any $\bar{x} \in M^n$. Let $\bar{c} := (c, \ldots, c) \in M^n$. Since

$$d_{\mathcal{M}^n}(\bar{x}, \bar{c}) = \max_{1 \leqslant j \leqslant n} d_{\mathcal{M}}(x_j, c) \leqslant \max_{1 \leqslant j \leqslant n} i = i,$$

it is at most an i-distance in $\mathcal{M}^n$. Consequently, $\bar{c}$ belongs to all the maximal (i)-sets in $\mathcal{M}^n$, and hence to their intersection as well! □

Definition 4.26 If a maximal (2)-set of an $n\star$-metric space possesses a vertex at a distance 1 from all the other vertices within it, then we refer to it as a *centre* of that maximal (2)-set.

Lemma 4.27 *Let $\mathcal{M}$ be a c-metric space with all of its different maximal (2)-sets being $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_l$, for some $l \in \mathbb{N}_+$. Suppose there exists a centre c_i of $\mathcal{K}_i$ for each $i \in \{1, 2, \ldots, l\}$. In that case, for every $n \in \mathbb{N}_+$ and every $\alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\}$ we have that $c^\alpha := (c_{\alpha(1)}, c_{\alpha(2)}, \ldots, c_{\alpha(n)})$ is a centre of the maximal (2)-set in $\mathcal{M}^n$ consisting of c^α together with all of its neighbours (in $\mathbf{G}_{\mathcal{M}^n}$). Additionally, all the maximal (2)-sets in $\mathcal{M}^n$ are of the proposed form, and there are exactly l^n of them.*

Proof. At first, we fix an $n \in \mathbb{N}_+$. Then, for all $\alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\}$, we denote the set of vertices adjacent to c^α in $\mathbf{G}_{\mathcal{M}^n}$, itself included, with $\mathcal{K}^\alpha$. Additionally, we notice that

$$\mathcal{K}^\alpha = \prod_{i=1}^{n} \mathcal{K}_{\alpha(i)}. \tag{4.1}$$

This is due to the following sequence of equivalent expressions:

$$\begin{aligned}
\bar{x} \in \mathcal{K}^\alpha \quad &\Leftrightarrow \quad d_{\mathcal{M}^n}(\bar{x}, c^\alpha) \leqslant 1 \\
&\Leftrightarrow \quad \max_{1 \leqslant i \leqslant n} d_{\mathcal{M}}(x_i, c_{\alpha(i)}) \leqslant 1 \\
&\Leftrightarrow \quad d_{\mathcal{M}}(x_i, c_{\alpha(i)}) \leqslant 1, \text{ for each } i \in \{1, 2, \ldots, n\} \\
&\Leftrightarrow \quad x_i \in \mathcal{K}_{\alpha(i)}, \text{ for each } i \in \{1, 2, \ldots, n\} \\
&\Leftrightarrow \quad \bar{x} \in \prod_{i=1}^{n} \mathcal{K}_{\alpha(i)}.
\end{aligned}$$

Evidently, we made use of the fact that $c_{\alpha(i)}$, as a centre, is connected to all the elements of $\mathcal{K}_{\alpha(i)}$ in $\mathbf{G}_{\mathcal{M}}$, for each $i \in \{1, 2, \ldots, n\}$. Clearly, $\mathcal{K}^\alpha$ is a subset of M^n, for each $\alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\}$ by definition. On the other hand, for each $\bar{x} \in M^n$ and each $i \in \{1, 2, \ldots, n\}$, there exists such $\beta_i \in \{1, 2, \ldots, l\}$ that $x_i \in K_{\beta_i}$, as each point of $\mathcal{M}$ belongs to at last one of its maximal (2)-sets. Thus $\bar{x} \in \mathcal{K}^\beta$, where $\beta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\} : i \mapsto b_i$. Altogether this implies that:

$$\bigcup_{\alpha:\{1,2,\ldots,n\}\to\{1,2,\ldots,l\}} \mathcal{K}^\alpha = M^n. \tag{4.2}$$

Now, owing to 4.1, for any two points $\bar{u}, \bar{v} \in \mathcal{K}^\alpha$ and any $i \in \{1, 2, \ldots, n\}$ we have that $\pi_i(\bar{u}) = u_i, \pi_i(\bar{v}) = v_i \in \mathcal{K}_{\alpha(i)}$ and consequently $d_{\mathcal{M}}(u_i, v_i)$ is at most a 2-distance in $\mathcal{M}$. What this implies is that

$$d_{\mathcal{M}^n}(\bar{u}, \bar{v}) = \max_{1 \leqslant i \leqslant n} d_{\mathcal{M}}(u_i, v_i)$$

is at most a 2-distance, too, but in $\mathcal{M}^n$. So, $\mathcal{K}^\alpha$ is a (2)-set!

Further on, we show that it is a maximal such. Take any $\bar{x} \in M^n \backslash \mathcal{K}^\alpha$. In case $\mathcal{K}^\alpha \dot{\cup} \{\bar{x}\}$ were still a (2)-set then $\bar{x}$ would be at most at a 2-distance from all the points of $\mathcal{K}^\alpha$. However, with respect to the maximum metric and 4.1, x_i is then at most at a 2-distance from ALL the rest of the points within $\mathcal{K}_{\alpha(i)}$, for every $i \in \{1, 2, \ldots, n\}$. Due to maximality of the (2)-sets

$\mathcal{K}_{\alpha(i)}$ in $\mathcal{M}$, for every $i \in \{1, 2, \ldots, n\}$, it holds that $x_i \in \mathcal{K}_{\alpha(i)}$! Nevertheless, it would then follow from 4.1 that $\bar{x} \in \mathcal{K}^\alpha$ which comes into collision with our initial choice of $\bar{x}$.

The fact that c^α is a centre of $\mathcal{K}^\alpha$, for each $\alpha : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\}$ follows trivially from the construction. So finally, we have shown that $\mathcal{K}^\alpha$ are in fact maximal (2)-sets within $\mathcal{M}^n$.

What remains to be proven, is our claim that all the maximal (2)-sets within $\mathcal{M}^n$ are of the proposed form. With that in view, let $\mathcal{K}^*$ be some maximal (2)-set in $\mathcal{M}^n$. Again, relying upon the maximum metric, we may freely claim that $\pi_i[\mathcal{K}^*]$ is a (2)-set in $\mathcal{M}$, for each $i \in \{1, 2, \ldots, n\}$. This is why there has to exist such a $\beta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\}$ for which $\pi_i[\mathcal{K}^*] \subseteq \mathcal{K}_{\beta(i)}$, for every $i \in \{1, 2, \ldots, n\}$. As a result, $\mathcal{K}^* \subseteq \mathcal{K}^\beta$, whereas due to the maximality of both of these (2)-sets they must be equal!

We have established that all the maximal (2)-sets in $\mathcal{M}^n$ do have a centre each which then uniquely determines them. Now, since these centres depended upon the variations of length n on the set of l elements of which there are l^n that is the upper limit of the number of the maximal (2)-sets in $\mathcal{M}^n$, as well. In order to prove the equality, we need to prove that no two maximal (2)-sets in $\mathcal{M}^n$, say $\mathcal{K}^\alpha$ and $\mathcal{K}^\beta$, are the same, when α and β are different variations. In that regard, take any such $\alpha, \beta : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, l\}$, that $\alpha \neq \beta$. Obviously, there exists an $i \in \{1, 2, \ldots, n\}$, for which $\alpha(i) \neq \beta(i)$. Consequently $\mathcal{K}_{\alpha(i)} \neq \mathcal{K}_{\beta(i)}$ and so their corresponding centers differ as well. If that were not the case, but $c_{\alpha(i)} = c_{\beta(i)}$ then $\mathcal{K}_{\alpha(i)} \cup \mathcal{K}_{\beta(i)}$ would still be a (2)-set, yet greater then either of the two it comprises, which contradicts their maximality! Again, due to the maximality of the aforementioned (2)-sets, there have to exist $x \in \mathcal{K}_{\alpha(i)}$ and $y \in \mathcal{K}_{\beta(i)}$ which are (at least) at a 3-distance. Furthermore, for $\bar{x} := (c_{\alpha(1)}, c_{\alpha(2)}, \ldots, c_{\alpha(i-1)}, x, c_{\alpha(i+1)}, \ldots, c_{\alpha(n)}) \in \mathcal{K}^\alpha$ and $\bar{y} := (c_{\beta(1)}, c_{\beta(2)}, \ldots, c_{\beta(i-1)}, y, c_{\beta(i+1)}, \ldots, c_{\beta(n)}) \in \mathcal{K}^\beta$ we have $d_{\mathcal{M}^n}(\bar{x}, \bar{y}) \geqslant d_{\mathcal{M}}(x, y) = 3$. What this implies is that $\bar{x}$ and $\bar{y}$ do not belong to the very same (maximal) (2)-set, and so $\mathcal{K}^\alpha \neq \mathcal{K}^\beta$, for sure. $\square$

Example 4.4 Proposition 4.27 gave us some insight into the appearance of the maximal (2)-sets of higher powers of a c-metric space. In Figure 12 the square of the metric space $\mathcal{M}$ from Example 4.2 is represented, with only its 1-distances shown. All of its maximal (2)-sets alongside with their corresponding centres, are coloured accordingly in the same way. Also the centres od $\mathcal{M}^2$ are labeled $c_1 := (w_1, w_1)$, $c_2 := (w_1, w_2)$, $c_3 := (w_2, w_1)$ and $c_4 := (w_2, w_2)$.

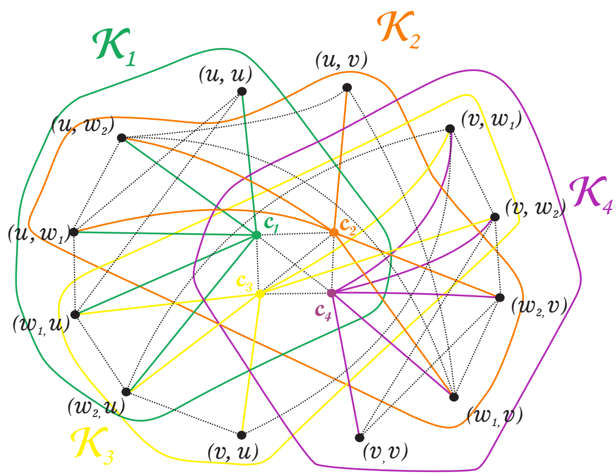

Figure 12: maximal (2)-sets with their centres in $\mathcal{M}^2$

Definition 4.28 A point of an $n\star$-metric space $\mathcal{M}$ is referred to as a *middle point* of a (k)-set in $\mathcal{M}$ in case it is at most at distance $\lceil \frac{k}{2} \rceil$ in $\mathbf{G}_{\mathcal{M}}$ from all the rest of the points of that (k)-set.

Proposition 4.29 *In a polymorphism-homogeneous c-metric space $\mathcal{M}$, there exists a middle point of any (k)-set in $\mathcal{M}$ where k is a positive integer* $\leqslant \operatorname{diam}(\mathbf{G}_{\mathcal{M}})$.

Proof. Let $u, v \in M$ be at the greatest k-distance a in $\mathcal{M}$, for some fixed $k \leqslant \operatorname{diam}(\mathbf{G}_{\mathcal{M}})$. Then, $d_{\mathcal{M}}(u, v) = a$ and there exists a path $uw_1w_2 \dots w_{k-1}v$ of length k in $\mathbf{G}_{\mathcal{M}}$, see Figure 13.

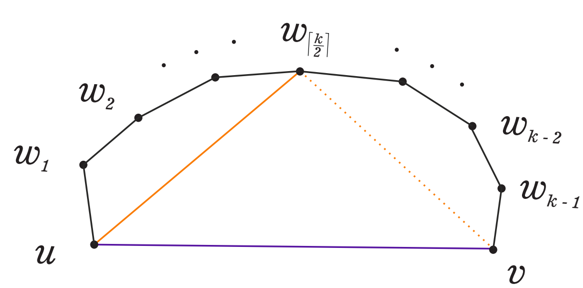

Figure 13: A path of length k connecting u and v

With all that in mind, $d_{\mathcal{M}}(u, w_{\lceil \frac{k}{2} \rceil})$ is precisely a $\lceil \frac{k}{2} \rceil$-distance, whereas $d_{\mathcal{M}}(w_{\lceil \frac{k}{2} \rceil}, v)$ is a $\lfloor \frac{k}{2} \rfloor$-distance, both in $\mathcal{M}$. Let $\mathcal{K} := \{x_1, x_2, \dots, x_n\}$ be some k-set in $\mathcal{M}$ and $|\mathcal{K}| =: n \geqslant 2$. Then the distance between any two points in

$$T := \{(u, v, v, \dots, v, v), (v, u, v, \dots, v, v), \dots, (v, v, v, \dots, v, u)\}$$

is exactly a in $\mathcal{M}^n$, whilst

$$\delta_{\mathcal{M}^n}((w_{\lceil \frac{k}{2} \rceil}, \dots, w_{\lceil \frac{k}{2} \rceil}), \bar{x}) = \left\lceil \frac{k}{2} \right\rceil,$$

for any $\bar{x} \in T$.

Due to the maximality of a the mapping $f : T \to \{(x_i, \dots, x_i) : i \in \{1, 2, \dots, n\}\}$ which maps $(v, v, \dots, v, u, v, \dots, v)$, with u as the i-th coordinate, to $(x_i, x_i, \dots, x_i)$, for every $1 \leqslant i \leqslant n$, is a well-defined local homomorphism. Hence, as $\mathcal{M}$ is polymorphism-homogenous, from Proposition 2.7, we have that $\mathcal{M}^n$ is homomorphism-homogenous, and so f can be expanded to an adequate endomorphism g of $\mathcal{M}^n$, as indicated in Figure 14. Consequently, there exists $(z_1, z_2, \dots, z_n) \in M^n$ such that

$$g((w_{\lceil \frac{k}{2} \rceil}, \dots, w_{\lceil \frac{k}{2} \rceil})) = (z_1, z_2, \dots, z_n),$$

and by the same token, from the non-expansiveness of g, for any $i \in \{1, 2, \dots, n\}$ we have:

$$\begin{aligned} & d_{\mathcal{M}^n}((x_i, x_i, \dots, x_i), (z_1, z_2, \dots, z_n)) \\ \leqslant \;& d_{\mathcal{M}^n}((v, v, \dots, v, u, v, \dots, v), (w_{\lceil \frac{k}{2} \rceil}, \dots, w_{\lceil \frac{k}{2} \rceil})) \\ \leqslant \;& \left\lceil \frac{k}{2} \right\rceil . \end{aligned}$$

Finally, for any $i \in \{1, 2, \dots, n\}$ it holds that

$$d_{\mathcal{M}}(x_i, z_1) \leqslant d_{\mathcal{M}^n}((x_i, x_i, \dots, x_i), (z_1, z_2, \dots, z_n)) \leqslant \left\lceil \frac{k}{2} \right\rceil,$$

making z_1 the desired middle point of $\mathcal{K}$. □

Corollary 4.30 *Let $\mathcal{M}$ be a polymorphism-homogenous c-metric space and k a positive integer. Any maximal (k)-set in $\mathcal{M}$ has a middle point belonging to it.*

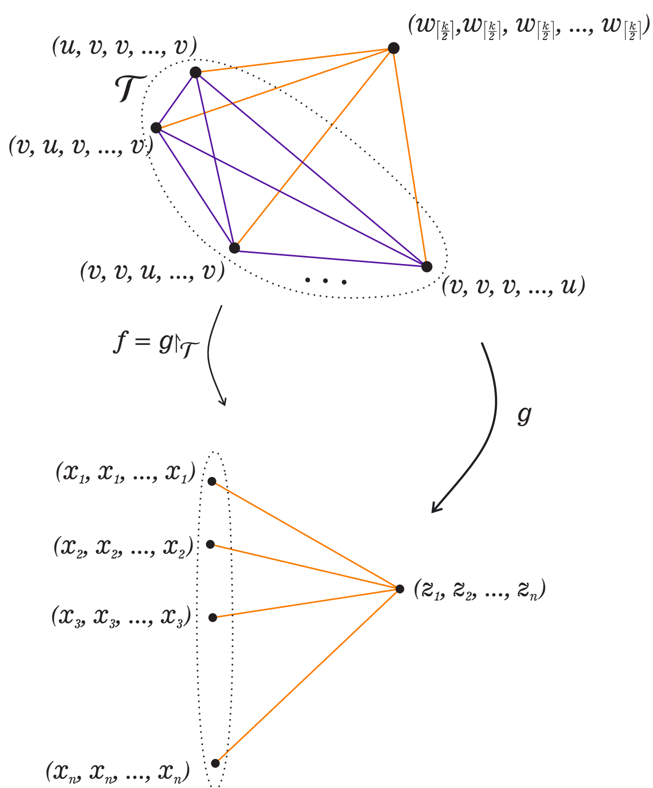

Figure 14: the existence of the vertex $(z_1, z_2, \ldots, z_n)$

Proof. At first, fix a maximal (k)-set $\mathcal{K}$ in $\mathcal{M}$. From Lemma 4.29, we trivially obtain a middle point $x \in M$ of $\mathcal{K}$. In other words, for any $y \in \mathcal{K}$

$$d_{\mathcal{M}}(x, y) \leqslant \left\lceil \frac{k}{2} \right\rceil \leqslant k.$$

Thus, $\mathcal{K} \cup \{x\}$ is still a (k)-set in $\mathcal{M}$. Finally, owing to the maximality of $\mathcal{K}$ we have $x \in \mathcal{K}$. □

Lemma 4.31 *Let $\mathcal{M}$ be a c-metric space and $\mathcal{L}$ a maximal (k)-set in $\mathcal{M}$, where k is an even positive integer. If $\mathcal{L}$ has a middle point x then*

$$\mathcal{L} = \{y \in M | \delta_{\mathcal{M}}(x, y) \leqslant \frac{k}{2}\}.$$

Proof. From the definition of x as a middle point it trivially follows that $\mathcal{L} \subseteq \{y \in M | \delta_{\mathcal{M}}(x, y) \leqslant \frac{k}{2}\}$. On the other hand, take any $u \in \mathcal{M}$ such that $\delta_{\mathcal{M}}(x, u) \leqslant \frac{k}{2}$ and any $v \in \mathcal{L}$. Bearing in mind that $\delta_{\mathcal{M}}(x, v) \leqslant \frac{k}{2}$, from the triangle inequality it follows that $\delta_{\mathcal{M}}(u, v) \leqslant 2 \cdot \frac{k}{2} = k$. Finally, from the maximality of $\mathcal{K}$ this implies that $u \in \mathcal{K}$. □

Remark. With the notion from the above Lemma, if $\mathcal{L}$ does contain two points which are at a k-distance in $\mathcal{M}$ then $\mathbf{G}_{\mathcal{M}}[\mathcal{L}]$ is a graph of diameter k, whose radius thus has to be $\geqslant \frac{k}{2}$. However, as $\varepsilon(x) \leqslant \frac{k}{2}$ in $\mathbf{G}_{\mathcal{M}}[\mathcal{L}]$ the equality is obtained.

Remark. Let $\mathcal{M}$ be a c-metric space and $i \in \mathbb{N}$. We say that an (i)-set $\mathcal{L}$ in $\mathcal{M}$ is within $\mathcal{K}$, if $\mathcal{L} \subseteq \mathcal{K}$. Further, $\mathcal{L}$ is a maximal (i)-set in $\mathcal{K}$ if it is within $\mathcal{K}$ and cannot be extended by any point in $\mathcal{K}$ to a strictly larger (i)-set within $\mathcal{K}$. Notice how a maximal (i)-set in $\mathcal{K}$ does not necessarily have to be a maximal (i)-set in $\mathcal{M}$. The former is however a (i)-set in $\mathcal{M}$, and as such can be extended to a maximal one in $\mathcal{M}$, which then evidently does not have to be fully within $\mathcal{K}$. This is why it is important to distinguish clearly between the (i)-sets which are maximal in $\mathcal{M}$, and those maximal in $\mathcal{K}$.

Corollary 4.32 *In a polymorphism-homogeneous c-metric space $\mathcal{M}$, the intersection of all the maximal $(\lceil \frac{k}{2} \rceil)$-sets within a maximal (k)-set in $\mathcal{M}$ is non-empty, for any positive integer k.*

Proof. Let $\mathcal{K}$ be a maximal (k)-set in $\mathcal{M}$. Corollary 4.30 provides us with a middle point x of $\mathcal{K}$. To put it differently, x belongs to all the maximal $(\lceil \frac{k}{2} \rceil)$-sets in $\mathcal{K}$, securing the non-emptiness of their intersection. $\square$

Lemma 4.33 *If there exists a point in the intersection of all the maximal (i)-sets of a c-metric space $\mathcal{M}$, then the intersection of all the maximal (i)-sets in $\mathcal{M}^n$ is again non-empty, for any $n \in \mathbb{N}_+$.*

Proof. Assume that for some fixed $i \in \mathbb{N}_+$ there exists some $c \in M$ which belongs to every single one of the maximal (i)-sets in $\mathcal{M}$. Now, fix $n \in \mathbb{N}_+$. It goes almost without saying that $\bar{c} := (c, c, \ldots, c) \in M^n$ belongs to every one of the maximal (i)-sets in $\mathcal{M}^n$. For any $\bar{x} := (x_1, x_2, \ldots, x_n) \in \mathcal{M}^n$,

$$d_{\mathcal{M}^n}(\bar{c}, \bar{x}) = \max_{1 \leqslant j \leqslant n} d_{\mathcal{M}}(c, x_j) = d_{\mathcal{M}}(c, x_{j^*}) \leqslant i,$$

for some $j^* \in \{1, 2, \ldots, n\}$. This leads to $d_{\mathcal{M}^n}(\bar{c}, \bar{x})$ being at most an i-distance in $\mathcal{M}^n$. $\square$

Lemma 4.34 *Let $\mathcal{M}$ be a c-metric space. If its skeleton has a vertex of eccentricity 1 then $\mathcal{M}$ is homomorphism-homogenous.*

Proof. We fix a point $u \in M$ of eccentricity 1. Clearly, u is connected to all the other vertices of $\mathbf{G}_{\mathcal{M}}$, thus being a universal vertex. Letting f be any local homomorphism of $\mathcal{M}$, we can easily expand it to $g \in \mathrm{End}(\mathcal{M})$, in the following way:

$$g(x) = \begin{cases} f(x), & x \in \mathrm{dom}(f) \\ u, & \text{otherwise} \end{cases}$$

In order to prove that g is a well-defined homomorphism, take any $x, y \in M$ and consider the following trivial cases:

- If $x, y \in \mathrm{dom}(f)$, then $d_{\mathcal{M}}(g(x), g(y)) = d_{\mathcal{M}}(f(x), f(y)) \leqslant d_{\mathcal{M}}(x, y)$, due to the non-expansiveness of f.
- If $x \in \mathrm{dom}(f)$ and $y \in M \setminus \mathrm{dom}(f)$, then

  $$d_{\mathcal{M}}(g(x), g(y)) = d_{\mathcal{M}}(f(x), u) \leqslant 1 \leqslant d_{\mathcal{M}}(x, y),$$

  as $x \neq y$ (since they belong to disjoint sets) and $\mathcal{M}$ is a normalised metric space.
- If $x, y \in M \setminus \mathrm{dom}(f)$, then $d_{\mathcal{M}}(g(x), g(y)) = d_{\mathcal{M}}(u, u) = 0 \leqslant d_{\mathcal{M}}(x, y)$.

This leads to the conclusion that g is a non-expansive mapping. Owing to the arbitrary choice of f, we may finally conclude that $\mathcal{M}$ is indeed homomorphism-homogeneous. $\square$

Lemma 4.35 *Let $\mathcal{M}$ be a c-metric space. If its skeleton has a vertex of eccentricity* 1 *then $\mathcal{M}$ is polymorphism-homogenous.*

Proof. Let $k \in \mathbb{N}_+$ and $u \in M$ a universal vertex in $\mathbf{G}_{\mathcal{M}}$. What we aim to show is that $\bar{u} := (u, u, \ldots, u) \in M^k$ is also a vertex of eccentricity 1, but clearly in the skeleton of $\mathcal{M}^k$. Hence, take an arbitrary $\bar{x} \in M^k$. Trivially:

$$d_{\mathcal{M}^k}(\bar{x}, \bar{u}) = \max_{1 \leqslant i \leqslant k} d_{\mathcal{M}}(x_i, u) \leqslant \max_{1 \leqslant i \leqslant k} 1 = 1,$$

which is exactly what we wanted. Further, applying Lemma 4.34 onto $\mathcal{M}^k$ we have that this c-metric space is homomorphism-homogeneous, for any $k \in \mathbb{N}_+$. Consequently, due to Proposition 2.7, $\mathcal{M}$ is polymorphism-homogeneous. $\square$

Lemma 4.36 *Any maximal* (2)*-set in a polymorphism-homogeneous c-metric space $\mathcal{M}$, induces a polymorphism-homogenous c-metric subspace of $\mathcal{M}$.*

Proof. Let $\mathcal{M}$ be a c-metric space and $\mathcal{K}$ a fixed maximal (2)-set. From Proposition 4.30 there exists a universal vertex u within that (2)-set. Once we make sure that $\mathbf{K} := (\mathcal{K}, d_{\mathcal{M}} \restriction_{\mathcal{K}})$ is indeed a c-metric space, the property of polymorphism-homogeneity shall follow directly from Lemma 4.35.

To put it differently, what remains is to show the non-existence of the $\star$-property within $\mathbf{K}$ alongside with the connectedness of $\mathbf{G}_{\mathbf{K}}$. The latter of the two is rather obvious, as for any two vertices $x, y \in M$ the path xuy connects them.

As $d_{\mathbf{K}} = d_{\mathcal{M} \restriction_{\mathcal{K}}}$, all the distances between any two points of $\mathcal{K}$ remain the same in both $\mathcal{M}$ and $\mathbf{K}$. That means that unless $\mathbf{K}$ is actually a (1)-set in $\mathcal{M}$, there exists at least one 2-distance b in $\mathbf{K}$. That implies the existence of $w_1, w_2 \in M$ such that $d_{\mathcal{M}}(w_1, w_2) = b$. Also, $d_{\mathcal{M}}(w_1, u) = 1$. We have, thus, found both a distance of length 1 and b in $\mathbf{K}$, whereas $1 < b \leqslant 2 * 1 = 2$, implying that $\mathbf{K}$ does not have the $\star$-property either. Returning to the exceptional case of $\mathbf{K}$ being a (1)-set, we have that (due to the maximality of it as a (2)-set) the very $\mathcal{M}$ would then be a (1)-set itself. However, as such it would not be a c-metric space which leads to a contradiction. $\square$

Corollary 4.37 *Any maximal* (2)*-set in a polymorphism-homogeneous c-metric space* $\mathcal{M}$ *possesses a centre, irrespective of* $\mathbf{G}_{\mathcal{M}}$*'s diameter.*

Proof. Combining the result of Corollary 4.30 for $d = 2$ with the very definition of a centre of some maximal (2)-set in $\mathcal{M}$ we straightforwardly obtain the statement of this corollary. □

Lemma 4.38 *To each of the maximal* (2)*-sets in a polymorphism-homogeneous c-metric space* $\mathcal{M}$ *corresponds a different centre.*

Proof. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be any two maximal (2)-sets in $\mathcal{M}$, with c_1 and c_2 being their corresponding centres. The existence of the latter two is due to Corollary 4.37. There also exist such $u \in \mathcal{K}_1$ and $v \in \mathcal{K}_2$ that $\delta_{\mathcal{M}}(u, v) = 3$, as $\mathcal{K}_1$ and $\mathcal{K}_2$ are two different maximal (2)-sets in $\mathcal{M}$. However, assuming the opposite, that $c_1 = c_2 =: c$ what we obtain is that

$$\delta_{\mathcal{M}}(u, v) \leqslant \delta_{\mathcal{M}}(u, c) + \delta_{\mathcal{M}}(c, v) = 1 + 1 = 2,$$

which leads to an obvious contradiction. □

Lemma 4.39 *Let* $\mathcal{M}$ *be a c-metric space satisfying the following two conditions:*

1) *every maximal* (4)*-set in* $\mathcal{M}$ *has a middle point, and*
2) *every maximal* (2)*-set in* $\mathcal{M}$ *has a centre.*

Additionally, let $\mathcal{K}$ *be a maximal* (4)*-set in* $\mathcal{M}$ *and* $\mathcal{L}$ *a maximal* (2)*-set in* $\mathcal{K}$*. Then* $\mathcal{L}$ *is a maximal* (2)*-set in* $\mathcal{M}$*, as well.*

Proof. Let $x \in \mathcal{K}$ be such that $\delta_{\mathcal{M}}(x, y) \leqslant 2$, for all $y \in \mathcal{K}$. Then it is the element of every maximal (2)-set in $\mathcal{K}$ — hence, $x \in \mathcal{L}$. Let $\widehat{\mathcal{L}}$ be an extension of $\mathcal{L}$ to a maximal (2)-set in $\mathcal{M}$, with its centre c. We have that $\delta_{\mathcal{M}}(c, x) \leqslant 1$. Additionally,

$$\forall y \in \widehat{\mathcal{L}} : \delta_{\mathcal{M}}(y, x) \leqslant 2.$$

From Lemma 4.31 we have that $\mathcal{K} = \{y \in M | \delta_{\mathcal{M}}(x, y) \leqslant 2\}$. Therefore, $\widehat{\mathcal{L}} \subseteq \mathcal{K}$. In conclusion $\widehat{\mathcal{L}} = \mathcal{L}$. □

Lemma 4.40 *Let $\mathcal{M}$ be a c-metric space satisfying the same two conditions from Lemma 4.39. Further, let $\mathcal{K}$ be its maximal* (4)*-set and $\overline{\mathcal{K}}$ a maximal* (3)*-set in $\mathcal{K}$. Then every maximal* (2)*-set $\mathcal{L}$ in $\mathcal{K}$ has each centre in $\overline{\mathcal{K}}$.*

Proof. Condition 1) together with Lemma 4.31 provides an $x \in \mathcal{K}$ such that $\mathcal{K} = \{y \in M | \delta_{\mathcal{M}}(x, y) \leqslant 2\}$. Let $\mathcal{L}$ be a maximal (2)-set in $\mathcal{K}$. Then $x \in \mathcal{L}$, and also $x \in \overline{\mathcal{K}}$. In particular, $\mathcal{L} \cap \overline{\mathcal{K}} \neq \emptyset$. Let, further, c be a centre of $\mathcal{L}$, and $u \in \overline{\mathcal{K}}$. Then $\delta_{\mathcal{M}}(c, x) \leqslant 1$ whereas $\delta_{\mathcal{M}}(x, u) \leqslant 2$, and so $\delta_{\mathcal{M}}(c, u) \leqslant 3$. Hence, $c \in \overline{\mathcal{K}}$. □

Corollary 4.41 *With the above notions, all the centres of all the maximal* (2)*-sets in $\mathcal{K}$ are contained within a maximal* (2)*-set in $\mathcal{K}$.*

Proof. Let $\mathcal{K}_1, \dots, \mathcal{K}_l$ be all the maximal (2)-sets in $\mathcal{K}$. Moreover, let x be a middle point of $\mathcal{K}$. Clearly, $x \in \bigcap_{i=1}^{l} \mathcal{K}_i$. Then, for every $i \in \{1, \dots, l\}$ and each centre c_i of $\mathcal{K}_i$ we have $\delta_{\mathcal{M}}(x, c_i) \leqslant 1$. Additionally, any two centres are at most at a 2-distance in $\mathcal{K}$, owing to x being their common neighbour. Further, letting C be the set of all the centres of all the maximal (2)-sets in $\mathcal{K}$. Then, $C \cup \{x\}$ is a (2)-set in $\mathcal{K}$. Hence, it is contained in some K_i, for $i \in \{1, \dots, l\}$. □

Lemma 4.42 *With the notions from Lemma 4.40, a maximal* (2)*-set in $\mathcal{K}$ that contains all the centres of all the maximal* (2)*-sets in $\mathcal{K}$ is contained in $\overline{\mathcal{K}}$.*

Proof. Let $\mathcal{L}$ be a maximal (2)-set in $\mathcal{K}$ that contains all the centres of all the maximal (2)-sets in $\mathcal{K}$. Further, let c be a centre of $\mathcal{L}$. Take $u \in \mathcal{L}$ and $v \in \overline{\mathcal{K}}$. Moreover, let $\mathcal{L}'$ be a maximal (2)-set in $\mathcal{K}$ containing v, and c' its centre. Then $\delta_{\mathcal{M}}(u, c) \leqslant 1$, $\delta_{\mathcal{M}}(c, c') = 1$ and $\delta_{\mathcal{M}}(c', v) \leqslant 1$ so $\delta_{\mathcal{M}}(u, v) \leqslant 3$. As a result, $\mathcal{L} \subseteq \overline{\mathcal{K}}$. □

Remark. As the choice of $\overline{\mathcal{K}}$ as a maximal (3)-set in $\mathcal{K}$ was arbitrary, as a consequence we have that $\mathcal{L}$ (from the above proof) is contained within the intersection of all the maximal (3)-sets in $\mathcal{K}$.

Lemma 4.43 *With the notions from above, for all $u, v \in \overline{\mathcal{K}}$ if $\delta_{\mathcal{M}}(u, v) = 3$, then there are centres c_i and c_j of the maximal* (2)*-sets in $\mathcal{K}$ such that uc_ic_jv is a path.*

Proof. Let $uxyv$ be a path. Then $\{u, x, y\}$ is a (2)-set. Hence, there is a maximal (2)-set $\mathcal{K}_i$ in $\mathcal{K}$, with centre c_i, that contains $\{u, x, y\}$. Thus uc_iyv is a path. Now, analogous to that we have that $\{c_i, y, v\}$ is again a (2)-set in $\mathcal{K}$. Therefore, there exists a maximal (2)-set $\mathcal{K}_j$ in $\mathcal{K}$ which contains it. Finally, uc_ic_jv is thus a path. □

Proposition 4.44 *Let $\mathcal{M}$ be a c-metric space with* $\operatorname{diam}(\mathbf{G}_{\mathcal{M}}) = 3$. *Let C be the set of all the centres of all the maximal (2)-sets in $\mathcal{M}$. Further let $H := \{c_1, \ldots, c_l\} \subset C$ be such that it induces a maximal complete subgraph in $\mathbf{G}_{\mathcal{M}}[C]$. Let $\mathcal{K}_1, \ldots, \mathcal{K}_l$ be the maximal (2)-sets induced by $c_1, \ldots, c_l$, respectively. Then $\bigcup_{i=1}^{n} \mathcal{K}_i = \mathcal{M}$.*

Proof. Let $C \backslash H = \{c_{l+1}, \ldots, c_n\}$. For every $i \in \{l+1, \ldots, n\}$ we denote by K_i the maximal (2)-set induced by c_i. By Corollary 4.41 C is contained in a maximal (2)-set in $\mathcal{M}$. A centre of that maximal (2)-set is of distance at most 1 from every element of C in $\mathbf{G}_{\mathcal{M}}$. Thus it must be an element of H. Without loss of generality, we may assume that c_1 is such a centre.

Assume that $\mathcal{M}$ is not covered by $\cup_{i=1}^{l} \mathcal{K}_i$. Fix such an $x \in \mathcal{M}$ which is not contained in any of the maximal (2)-sets with centres in H. Clearly, there exists a $j \in \{1, 2, \ldots, n\}$ such that $x \in \mathcal{K}_j$. Moreover, such j must be greater then l, as $c_j \notin H$. What we will now show is that there exists a $k \in \{1, 2, \ldots, l\}$ such that $\delta_{\mathcal{M}}(c_k, x) = 3$. Suppose to the contrary, that $\delta_{\mathcal{M}}(c_k, x) = 2$ for all $k \in \{1, 2, \ldots, l\}$. In that case $H \dot{\cup} \{c_j, x\}$ would be a (2)-set in $\mathcal{M}$ and thus contained within $\mathcal{K}_p$, for some $p \in \{1, 2, \ldots, n\}$. However, c_p would then be at distance at most 1 in $\mathbf{G}_{\mathcal{M}}$ from all the vertices in H. Due to the maximality of the latter, we would have $c_p \in H$. Thus, $\delta_{\mathcal{M}}(c_p, x) = 2$. But, this is a contradiction to $\delta_{\mathcal{M}}(c_p, x) \leqslant 1$. Therefore, there exists a $k \in \{1, 2, \ldots, l\}$ such that $\delta_{\mathcal{M}}(c_k, x) = 3$. That immediately implies $\delta_{\mathcal{M}}(c_k, c_j) = 2$, see Figure 15.

Let all the maximal (2)-sets containing x be $\mathcal{K}_{i_1}, \mathcal{K}_{i_2}, \ldots, \mathcal{K}_{i_m}$. Trivially $j \in \{i_1, i_2, \ldots, i_m\}$. Obviously, none of their centres belong to H. Furthermore, we define $P \subseteq H$ such that it contains all the points from H which are precisely at a 2-distance from x. Evidently, P is not empty as c_1 belongs to it. Now, notice how $P \cup \{c_{i_1}, c_{i_2}, \ldots, c_{i_m}, x\}$ is a (2)-set and hence contained in some $\mathcal{K}_p$, for some $p \in \{1, 2, \ldots, n\}$. As $x \in \mathcal{K}_p$ then $p \in \{i_1, i_2, \ldots, i_m\}$. Without loss of generality we may assume that $p = i_1$. As $\delta_{\mathcal{M}}(c_k, c_{i_1}) = 2$, there exists such a $y \in \mathcal{K}_k \backslash \mathcal{K}_{i_1}$ that $\delta_{\mathcal{M}}(c_{i_1}, y) = 3$, see Figure 16. If such a y were not to exist, but $\delta_{\mathcal{M}}(c_{i_1}, y) \leqslant 2$, for all $y \in \mathcal{K}_k$, then $\{c_{i_1}\} \dot{\cup} \mathcal{K}_k$ would be a (2)-set. From the maximality of $\mathcal{K}_k$ as a (2)-set in $\mathcal{M}$ we would

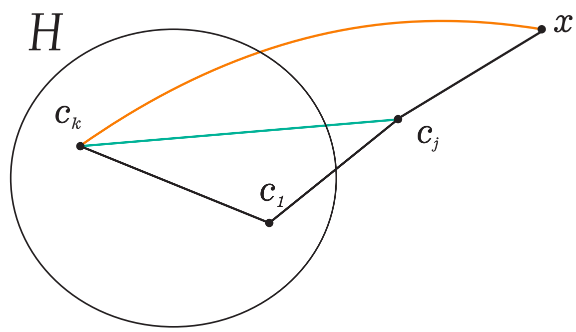

Figure 15: $x \in \mathcal{M}$ at a (3)-distance from a vertex in H

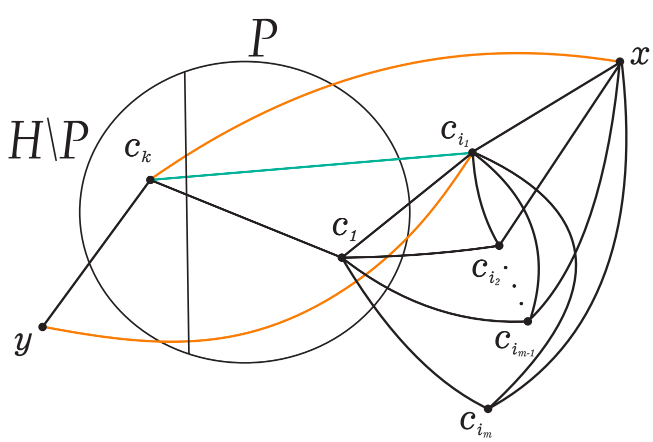

Figure 16: $\delta_{\mathcal{M}}(c_{i_1}, y) = 3$

have $c_{i_1} \in \mathcal{K}_k$, which is impossible (because $\delta_{\mathcal{M}}(c_k, c_{i_1}) = 2$). Additionally, we have that none of the centres $c_{i_1}, \ldots, c_{i_m}$ is at a distance 1 in $\mathbf{G}_{\mathcal{M}}$ from y, since it would then be a common neighbour of c_{i_1} and y in $\mathbf{G}_{\mathcal{M}}$, but $\delta_{\mathcal{M}}(y, c_{i_1}) = 3$.

Further, we consider the distance between x and y in $\mathbf{G}_{\mathcal{M}}$. Clearly, $\delta_{\mathcal{M}}(x, y) \leqslant 3$.

(i) If $\delta_{\mathcal{M}}(x, y) = 1$, then we reach a contradiction with

$$3 = \delta_{\mathcal{M}}(y, c_{i_1}) \leqslant \delta_{\mathcal{M}}(y, x) + \delta_{\mathcal{M}}(x, c_{i_1}) = 1 + 1 = 2.$$

(ii) If $\delta_{\mathcal{M}}(x, y) = 2$, then there exists a $q \in \{1, 2, \ldots, m\}$ such that $\{x, y\} \subseteq \mathcal{K}_{i_q}$. However, as for any such q, $\delta_{\mathcal{M}}(c_{i_q}, y) \neq 1$ this is clearly impossible.

(iii) If $\delta_{\mathcal{M}}(x, y) = 3$, then from Lemma 4.43 there exist such $s \in \{1, 2, \ldots, n\}$ and $q \in \{1, 2, \ldots, m\}$ that $x c_{i_q} c_s y$ is a path, see Figure 17. We show

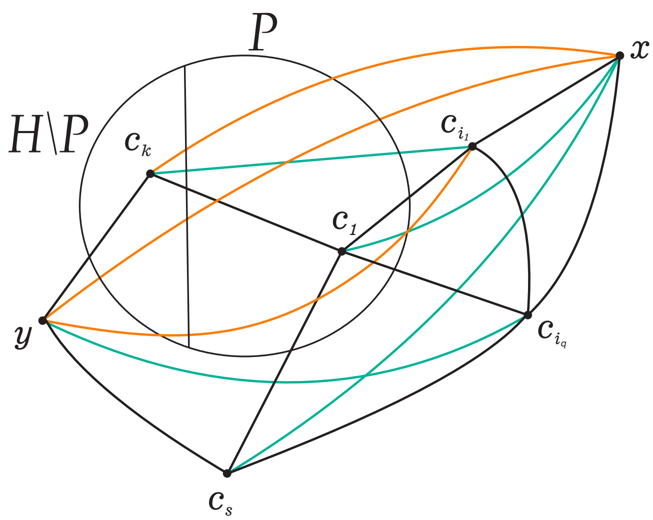

Figure 17: $\delta_{\mathcal{M}}(x, y) = 3$

that $\delta_{\mathcal{M}}(y, c_{i_q}) = 2$. Obviously, $\delta_{\mathcal{M}}(y, c_{i_q}) \leqslant 2$, since c_s is then a common neighbour of both y and c_{i_q}. Further, since $\delta_{\mathcal{M}}(y, c_{i_1}) = 3$ then $y c_{i_q} c_{i_1}$ cannot be a path of length 2, meaning that $\delta_{\mathcal{M}}(y, c_{i_q}) \neq 1$. With all that in mind, $H \cup \{y, c_{i_q}\}$ is then a (2)-set. Therefore, there must exist some $r \in \{1, 2, \ldots, n\}$ that $H \cup \{y, c_{i_q}\} \subseteq \mathcal{K}_r$. Owing to the maximality of H, this implies that $c_r \in H$ and so $\delta_{\mathcal{M}}(c_r, x) \neq 1$, see Figure 18. Moreover, c_r must be in P, as

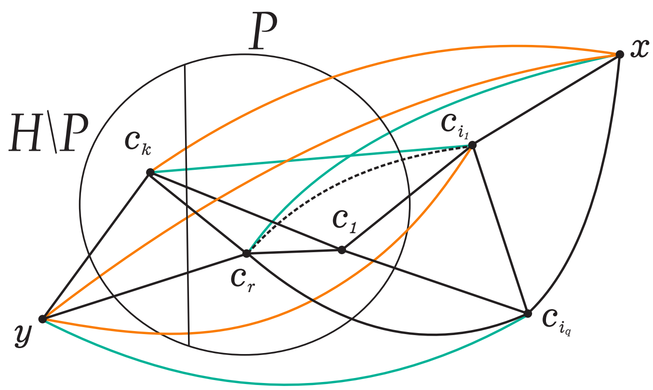

Figure 18: $c_r \in P$

$$\delta_{\mathcal{M}}(x, c_r) \leqslant \delta_{\mathcal{M}}(x, c_{i_q}) + \delta_{\mathcal{M}}(c_{i_q}, c_r) = 1 + 1 = 2.$$

Then $\delta_{\mathcal{M}}(c_{i_1}, c_r) = 1$, which leads to

$$3 = \delta_{\mathcal{M}}(y, c_{i_1}) \leqslant \delta_{\mathcal{M}}(y, c_r) + \delta_{\mathcal{M}}(c_r, c_{i_1}) = 1 + 1 = 2,$$

which evidentally is not true.

Finally, what we have proven is that the maximal (2)-sets with their centres in H do cover the whole space $\mathcal{M}$.

□

Corollary 4.45 *Let $\mathcal{M}$ be a c-metric space with* $\operatorname{diam}(\mathbf{G}_{\mathcal{M}}) = 3$ *such that it fulfills all the premises of Proposition 4.44. Let c be a centre of a maximal (2)-set in $\mathcal{M}$, and let $u \in \mathcal{M}$. Then $\delta_{\mathcal{M}}(c, u) \leqslant 2$.*

Proof. Let C be the set of all centres of all the maximal (2)-sets in $\mathcal{M}$ and D a maximal complete subgraph of $\mathbf{G}_{\mathcal{M}}[C]$ containing c. By Proposition 4.44 we have that there exists a $v \in D$ such that $\delta_{\mathcal{M}}(u, v) \leqslant 1$. Hence, $\delta_{\mathcal{M}}(u, c) \leqslant 2$. □

Lemma 4.46 *Let $\mathcal{M}$ be a polymorphism-homogeneous c-metric space, then* $\operatorname{diam}(\mathbf{G}_{\mathcal{M}}) \geqslant 2$.

Proof. If $\operatorname{diam}(\mathbf{G}_{\mathcal{M}}) \leqslant 1$ then there would exist an edge between any two of n vertices of $\mathbf{G}_{\mathcal{M}}$. That would imply that $\mathbf{G}_{\mathcal{M}} \cong K_n$. However, as

$\mathrm{im}(d_{\mathcal{M}}) \subseteq \{0,1\}$ then $\mathcal{M}$ would have the $\star$-property, which is not the case. □

Theorem 4.47 *A c-metric space $\mathcal{M}$ with $\mathrm{diam}(\mathbf{G}_{\mathcal{M}}) = 2$ is polymorphism-homogenous if and only if its skeleton possess a universal vertex.*

Proof. ($\Longrightarrow$) It suffices to notice how $\mathcal{M}$ on its own is a maximal (2)-set in $\mathcal{M}$. Thus, due to its polymorphism-homogeneity Corollary 4.37 implies that $\mathcal{M}$ possesses a centre, which in this case is at the same time a universal vertex of its skeleton. ($\Longleftarrow$) The opposite implication is a trivial consequence of Lemma 4.35, for the special case of $\mathcal{M}$ having a skeleton of diameter 2. □

Proposition 4.48 *Let $\mathcal{M}$ be a polymorphism-homogeneous c-metric space with $\mathrm{diam}(\mathbf{G}_{\mathcal{M}}) = 3$. Further, let a be the greatest (3)-distance in it and b the shortest 2-distance over a. Then, for any $T \subset M$ there must exist a point in $\mathcal{M}$ which is at a distance at most b in $\mathcal{M}$ from each of the points from T.*

Proof. The proof is analogous to the one of Proposition 4.29, with the exception of choosing not just any path uw_1w_2v of length 3, but the one for which the (2)-distances over a in $\mathcal{M}$ are the shortest, i. e. which are b. The existence of such a path is secured by Lemma 4.18, for $i = 3$. □

Theorem 4.49 *A c-metric space $\mathcal{M}$ with $\mathrm{diam}(\mathbf{G}_{\mathcal{M}}) = 3$ is polymorphism-homogeneous whenever all the following conditions are satisfied:*

1) *for all $u, v, s, t \in M$ if $\delta_{\mathcal{M}}(s,t) < \delta_{\mathcal{M}}(u,v)$ then $d_{\mathcal{M}}(s,t) < d_{\mathcal{M}}(u,v)$;*
2) *every maximal (2)-set in $\mathcal{M}$ contains a centre;*
3) *the intersection of all the maximal (2)-sets in $\mathcal{M}$ is non-empty, and*
4) *there exists only one 2-distance.*

Proof. All the conditions inflicted upon $\mathcal{M}$ remain valid for all of its direct powers, just the same due to Propositions 4.22, 4.27 and 4.24. Consequently, once we prove that $\mathcal{M}$ is homomorphism-homogeneous, every one of its direct powers will be, as well. On the whole, Proposition 2.7 will provide polymorphism-homogeneity of $\mathcal{M}$.

We begin by showing that $\mathcal{M}$ possesses the one-point extension property, the direct consequence of which (by Proposition 2.9) is the homomorphism-homogeneity of the metric space in point. Take any local homomorphism f of $\mathcal{M}$, with the domain $T = \{x_1, x_2, \ldots, x_k\}$. Let further $z \in M \backslash T$. We then extend f to a homomorphism $g : T \cup \{z\} \mapsto M$, in the way described below.

(1) If $u \in T$ then $g(u) := f(u)$, of course, and g is a well-defined extension.

(2) If z is at a 1-distance in $\mathcal{M}$ from some point in T, then let T_z be the set of all the neighbours of z from T in $\mathbf{G}_{\mathcal{M}}$. Note that the distance of z from any element of $T \backslash T_z$ in $\mathbf{G}_{\mathcal{M}}$ is at least 2. Evidently, T_z is a (2)-set in $\mathcal{M}$. Thus $f[T_z]$ is a (2)-set in $\mathcal{M}$, too (owing to the non-expansiveness of f). Let c be a centre of a maximal (2)-set containing $f[T_z]$. Then c is of distance at most 2 from any element of $f[T \backslash T_z]$ in $\mathbf{G}_{\mathcal{M}}$. Define $g(z) := c$. Since $\mathcal{M}$ has just one 2-distance, g is non-expansive.

(3) If $\delta_{\mathcal{M}}(x_i, z) \geqslant 2$ for all $i \in \{1, 2, \ldots, k\}$, then simply let $g(z) := c^*$, where c^* is in the intersection of all the maximal (2)-sets in $\mathcal{M}$. Since $\delta_{\mathcal{M}}(y, c^*) \leqslant 2$, for all $y \in f[T]$, and since $\mathcal{M}$ has just one 2-distance, it follows that g is non-expansive.

Now it is clear that g is a homomorphic extension of f, and as a result $\mathcal{M}$ has the one-point extension property, which is just what we needed. □

Example 4.5 There exist c-metric spaces with exactly one 2-distance, which are not polymorphism-homogeneous. One such example is shown in Figure 19. Notice how not all of its maximal (2)-sets possess a centre. Clearly,

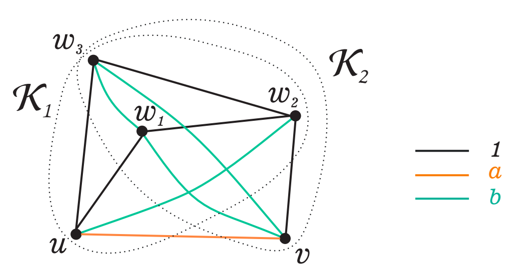

Figure 19: A non-PH metric space

w_2 is the centre of $\mathcal{K}_2$, whereas no point in $\mathcal{K}_1$ could be a centre of that particular maximal (2)-set.

Example 4.6 In Figure 20 we see an example of a c-metric space $\mathcal{M}$ with one 3-distance a and two different 2-distances b and c within it. Although it does satisfy the necessary condition imposed by Proposition 4.18, it does not comply with Corollary 4.20, both for $i = 3$. This sudden inappropriate variety of 2-distances in $\mathcal{M}$ make it non-homomorphism-homogenous! The same conclusion could have been reached directly, by finding a local

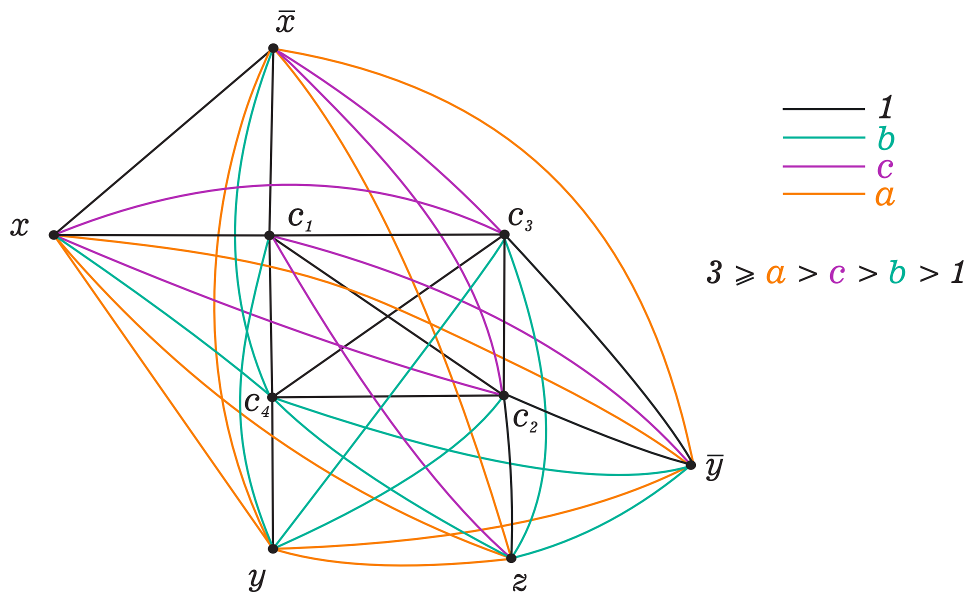

Figure 20: A non-HH metric space with two different 2-distances

homomorphism of $\mathcal{M}$ which can not be expanded to an endomorphism of $\mathcal{M}$. One such is f defined so as to map x onto $\bar{x}$ and y onto $\bar{y}$. Namely, if it could be expanded to $g \in \text{End}(\mathcal{M})$ then $g(c_4) \in \{c_2, c_3, \bar{y}\}$, due to the non-expansiveness of g in combination with the fact that $d_{\mathcal{M}}(y, c_4) = 1$. However, that is not all! It would also have to satisfy the following:

$$d_{\mathcal{M}}(\bar{x}, g(c_4)) = d_{\mathcal{M}}(g(x), g(c_4)) \leqslant d_{\mathcal{M}}(x, c_4) = b.$$

Simply putting all the three candidates to the test we reach a clear contradiction with the previous inequality, as $d_{\mathcal{M}}(\bar{x}, c_2) = d_{\mathcal{M}}(\bar{x}, c_3) = c > b$ and $d_{\mathcal{M}}(\bar{x}, \bar{y}) = a > b$!

4.2.2 $n\star$-metric spaces with disconnected skeletons

We now finally turn to the general case of $n\star$-metric spaces whose skeleton is not a connected graph.

Definition 4.50 Let $\mathcal{M}$ be a $n\star$-metric space and $P \subseteq M$ be such that $\mathbf{G_P} := \mathbf{G}_{\mathcal{M}}[P]$ is a connected component of $\mathbf{G}_{\mathcal{M}}$. The metric subspace $\mathbf{P}$ induced by P in $\mathcal{M}$ is then referred to as an $\mathcal{M}$-*connected component*.

Lemma 4.51 *Any endomorphism on an $n\star$-metric space $\mathcal{M}$, maps an $\mathcal{M}$-connected component into some $\mathcal{M}$-connected component.*

Proof. Let $P \subset M$ be such that $\mathbf{P} := \mathcal{M}[P]$ is an $\mathcal{M}$-connected component. Take any $f \in \mathrm{End}(\mathcal{M})$. By definition, $\mathbf{G_P}$ is a connected component of $\mathbf{G}_{\mathcal{M}}$. From Lemma 4.11 we have that $f_{\restriction \mathbf{G_P}}$ is a local homomorphism of the graph $\mathbf{G}_{\mathcal{M}}$. Consequently, there exists some $Q \subseteq M$ for which $\mathbf{G_Q}$ is a connected component of the skeleton of $\mathcal{M}$, such that $f[P] = f_{\restriction \mathbf{P}}[P] = f_{\restriction \mathbf{G_P}}[P] \subseteq Q$. Finally, since f was an endomorphism of $\mathcal{M}$ for $\mathbf{Q} := \mathcal{M}[Q]$ we have that $f[\mathbf{P}] \subset \mathbf{Q}$ whereas, by definition, $\mathbf{Q}$ is an $\mathcal{M}$-connected component. $\square$

Combining the results of Lemma 4.51 and Corollary 4.16 we obtain the following:

Corollary 4.52 *Let $\mathcal{M}$ be an $n\star$-metric space and $\mathbf{P}$ some $\mathcal{M}$-connected component whose skeleton is of diameter i. Then, for any $f \in \mathrm{End}(\mathcal{M})$, $f[P]$ is a (j)-set within some $\mathcal{M}$-connected component, where $j \leqslant i$.*

Lemma 4.53 *Let $\mathcal{M}$ be an $n\star$-metric space. If there exists a non-homomorphism-homogenous $\mathcal{M}$-connected component, then $\mathcal{M}$ is not homomorphism-homogenous either.*

Proof. Assuming the opposite, let $\mathbf{P} = (P, d_{\mathcal{M}} \restriction_P)$ be a non-homomorphism-homogenous $\mathcal{M}$-connected component of the homomorphism-homogeneous $\mathcal{M}$. Then, let f be a local homomorphism on $\mathbf{P}$ which cannot be extended to an endomorphism of $\mathbf{P}$. Since $\mathbf{P}$ is a metric subspace of $\mathcal{M}$ then trivially $\bar{f} : \mathrm{dom}(f) \to M : x \mapsto f(x)$ is a well-defined local homomorphism of $\mathcal{M}$. Due to homomorphism-homogeneity of $\mathcal{M}$, there has to exist a homomorphic extension $\bar{g} \in \mathrm{End}(\mathcal{M})$ of $\bar{f}$. As $\mathrm{dom}(\bar{f}) = \mathrm{dom}(f) \subseteq P$ then $\bar{g}_{\restriction \mathrm{dom}(f)} = \bar{f}_{\restriction \mathrm{dom}(f)} = f$, so $\bar{g}[\mathrm{dom}(f)] = f[\mathrm{dom}(f)] = \mathrm{im}(f) \subseteq P$. Consequently, relying upon Lemma 4.51 we have that $\bar{g}[\mathbf{P}] \subseteq \mathbf{P}$. That in turn implies that $\bar{g} \restriction_{\mathbf{P}} \in \mathrm{End}(\mathbf{P})$. However, since $(\bar{g} \restriction_{\mathbf{P}}) \restriction_{\mathrm{dom}(f)} = \bar{g}_{\restriction \mathrm{dom}(f)} = f$ we have that $\bar{g} \restriction_{\mathbf{P}}$ is in fact a homomorphic extension of f which contradicts the non-homomorphism-homogeneity of $\mathbf{P}$. $\square$

To put it differently,

If $\mathcal{M}$ is a homomorphism-homogenous $n\star$-metric space, then every $\mathcal{M}$-connected component is homomorphism-homogeneous as well.

Lemma 4.54 *Let* $\mathbf{P}$ *be an* $\mathcal{M}$*-connected component, where* $\mathcal{M}$ *is an* $n\star$*-metric space. For any* $n \in \mathbb{N}_+$ *we have that* $\mathbf{P}^n$ *is then an* $\mathcal{M}^n$*-connected component.*

Proof. Fix some positive integer n.

At first we show that $\mathbf{G}_{\mathbf{P}^n}$ is connected. Take any two points $\bar{u}, \bar{v} \in P^n$. Clearly for each $i \in \{1, 2, \ldots, n\}$ both u_i and v_i belong to P. Since $\mathbf{G}_\mathbf{P}$ is connected, for each $i \in \{1, 2, \ldots, n\}$ there have to exist some $w_1^i, w_2^i, \ldots, w_{l_i}^i \in P$ such that $u_i w_1^i w_2^i \ldots w_{l_i}^i v_i$ is a path (within $\mathbf{P}$) connecting u_i and v_i, and $l_i \in \mathbb{N}_+$. Then define $l := \max\limits_{1 \leqslant i \leqslant n} l_i$. With all that in mind, for each $j \in \{1, 2, \ldots, l\}$, we define a point $\bar{w}_j := (w_j^1, w_j^2, \ldots, w_j^n)$, where $w_j^i := v_i$, whenever $j > l_i$, for $i \in \{1, 2, \ldots, n\}$. It is now fairly easy to prove that $\bar{u}\bar{w}_1\bar{w}_2 \ldots \bar{w}_l\bar{v}$ is a path in $\mathbf{G}_{\mathbf{P}^n}$ connecting $\bar{u}$ and $\bar{v}$ thus proving $\mathbf{G}_{\mathbf{P}^n}$ to be connected.

Secondly, we prove that $\mathbf{G}_{\mathbf{P}^n}$ is maximal, in the sense that there exists no point in $M^n \backslash P^n$ connected to some point of $\mathbf{P}^n$. In fact, it suffices to show that there is no $\bar{u} \in P^n$ at a distance precisely 1 from any of the point in $M^n \backslash P^n$. Assuming the opposite, let $\bar{u} \in P^n$ and $\bar{x} \in M^n \backslash P^n$ that

$$1 = d_{\mathcal{M}^n}(\bar{x}, \bar{u}) = \max_{1 \leqslant i \leqslant n} d_\mathcal{M}(x_i, u_i).$$

For each $i \in \{1, 2, \ldots, n\}$ we now have that $d_\mathcal{M}(x_i, u_i) \leqslant 1$. To put it differently, x_i is either equal to u_i or at a distance 1 from it. Bearing in mind that $\mathbf{G}_\mathbf{P}$ is connected, that all results in $x_i \in P$, as $u_i \in P$, for all $i \in \{1, 2, \ldots, n\}$. However, $\bar{x}$ would then have to belong to P^n, with all its coordinates being in P, which contradicts the very choice of that point!

On the whole, we have shown that $\mathbf{G}_{\mathbf{P}^n}$ is a connected component of $\mathbf{G}_\mathcal{M}$ thus proving $\mathbf{P}^n$ to be an $\mathcal{M}^n$-connected component. □

Corollary 4.55 *Let* $\mathcal{M}$ *be an* $n\star$*-metric space. If any* $\mathcal{M}$*-connected component is not polymorphism-homogenous then* $\mathcal{M}$ *is not polymorphism-homogeneous either.*

Proof. Let $\mathbf{P}$ be some non-polymorphism-homogenous $\mathcal{M}$-connected component. Having Proposition 2.7 in mind, there exists some $k \in \mathbb{N}_+$ for which $\mathbf{P}^k$ is not homomorphism-homogenous. Now, from Lemma 4.54 we have that $\mathbf{P}^k$ is itself a $\mathcal{M}^k$-connected component implying that $\mathcal{M}^k$ is not homomorphism-homogenous, due to Lemma 4.53. However, applying

Proposition 2.7 once again, we have that now $\mathcal{M}$ cannot be polymorphism-homogenous! ☐

Example 4.7 The opposite implication, however, does not hold for neither Lemma 4.53 nor for Corollary 4.55! A counterexample to both is provided in Figure 21, as a sketch of a metric space $\mathcal{M}$ without the $\star$-property yet with two $\mathcal{M}$-connected components. The distances between these two $\mathcal{M}$-components have been omitted from the sketch so as to simplify it. Clearly, a_1 and a_2 represent the 3-distances, whereas b_1 and b_2 are the (minimal) 2-distances over them, respectively. $u_1u_2u_3u_4$ and $v_1v_2v_3v_4$ are paths of length 3, and we fix $a_1 \geqslant a_2$ and $b_1 < b_2$. From Corollary 4.19 for $i = 3$, it trivially follows that $\mathcal{M}$ in neither homomorphism-homogeneous, nor consequently polymorphism-homogenous. The fact that the connected components are themselves polymorphism-homogenous (and thus homomorphism-homogenous as well) follows from Theorem 4.49.

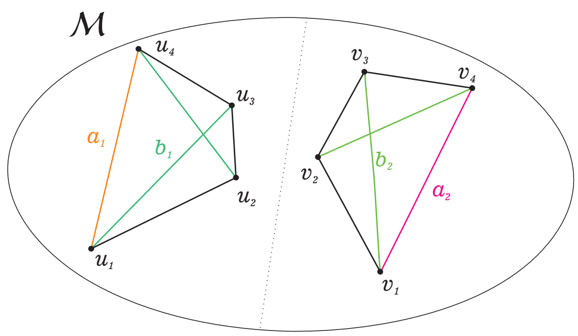

Figure 21: A non-PH c-metric space $\mathcal{M}$ with 2 PH connected components

Lemma 4.56 *The distance between the points belonging to different $\mathcal{M}$-connected components of a homomorphism-homogenous $n\star$-metric space $\mathcal{M}$, is necessarily greater than any distance between any two points of $\mathcal{M}$ within the same $\mathcal{M}$-connected component.*

Proof. Assuming the opposite holds, let $u_1, v_1 \in M$ be some points belonging to the same, and $u_2, v_2 \in M$ to two different $\mathcal{M}$-connected components, satisfying the inequality

$$d_{\mathcal{M}}(u_1, v_1) \geqslant d_{\mathcal{M}}(u_2, v_2).$$

Then, the mapping f defined so as to map u_1 to u_2 and v_1 to v_2 is obviously a well-defined local homomorphism of $\mathcal{M}$. However it cannot be expanded to a global homomorphism of $\mathcal{M}$, as the images of u_1 and v_1 through it would have to belong to the same $\mathcal{M}$-connected component, as per Lemma 4.51, which cannot be the case as those would necessarily be $f(u_1) = u_2$ and $f(v_1) = v_2$, respectively. $\square$

4.3 Metrically polymorphism-homogeneous connected graphs

Every connected graph gives rise to a metric space when the set of vertices is accompanied by the standard graph metric, mentioned in Section 2.3. Thus we give a fairly natural definition below.

Definition 4.57 A connected graph is *metrically polymorphism-homogeneous* if it is polymorphism-homogeneous when considered as a metric space in the graph metric.

Lemma 4.58 *When* $\mathbf{H}$ *is a finite connected graph of diameter* $\geqslant 2$ *then* $\overline{\mathbf{H}} := (V(\mathbf{H}), d_{\mathbf{H}})$ *is a c-metric space with only one* 2*-distance. Additionally,* $\mathbf{G}_{\overline{\mathbf{H}}} = \mathbf{H}$ *and* $\delta_{\overline{\mathbf{H}}} = d_{\mathbf{H}}$.

Proof. Since $\text{diam}(\mathbf{H}) \geqslant 2$ there exists a path of length at least 2 in $\mathbf{H}$. Let u, c, v be three consecutive vertices of that path, in that order. Then $d_{\mathbf{H}}(u, c) = 1$ and $d_{\mathbf{H}}(u, v) = 2$. Therefore, $\overline{\mathbf{H}}$ is a metric space without the $\star$-property. Further, as $\mathbf{H}$ is finite then $\overline{\mathbf{H}}$ is finite, as well and 1 is trivially the smallest nonzero distance in it. Consequently, $\overline{\mathbf{H}}$ is an $n\star$-metric space. It goes almost without saying that the skeleton of $\overline{\mathbf{H}}$ is truly $\mathbf{H}$. This is because:

$$(u, v) \in \mathbf{G}_{\overline{\mathbf{H}}} \leftrightarrow d_{\overline{\mathbf{H}}}(u, v) = d_{\mathbf{H}}(u, v) \leqslant 1 \leftrightarrow (u, v) \in E(\mathbf{H}).$$

Therefore, $\delta_{\overline{\mathbf{H}}} = d_{\mathbf{H}}$. Finally, as $\mathbf{H}$ is connected, then so is the skeleton of $\overline{\mathbf{H}}$ and hence $\overline{\mathbf{H}}$ is a c-metric space. Obviously, 2 is the only 2-distance in $\overline{\mathbf{H}}$. $\square$

Lemma 4.59 *When* $\mathbf{H}$ *is a finite connected graph of diameter* $\leqslant 1$ *then it is metrically polymorphism-homogeneous.*

Proof. Let $n := |V(\mathbf{H})|$. Clearly, $\mathbf{H} \cong K_n$ and so $(V(\mathbf{H}), d_{\mathbf{H}})$ is a finite $\star$-metric space. Theorem 4.9 implies that $\mathbf{H}$ is then metrically polymorphism-homogeneous. $\square$

We may now finally give a full classification of finite metrically polymorphism-homogenous graphs, up to diameter 3.

Theorem 4.60 *Let* $\mathbf{H}$ *be a finite connected graph of diameter at most* 3 *and let* $\overline{\mathbf{H}} := (V(\mathbf{H}), d_{\mathbf{H}})$*. Then* $\mathbf{H}$ *is metrically polymorphism-homogeneous if, and only if, either* $\operatorname{diam}(\mathbf{H}) \leqslant 1$ *or the following set of conditions are satisfied:*

1) $\overline{\mathbf{H}}$ *is a c-metric space,*

2) every maximal (2)*-set in* $\overline{\mathbf{H}}$ *contains a centre, and*

3) the intersection of all the maximal (2)*-sets in* $\overline{\mathbf{H}}$ *is non-empty.*

Proof. Let $\overline{\mathbf{H}}$ be a polymorphism-homogeneous metric space. Assume that $\operatorname{diam}(\mathbf{H}) \in \{2, 3\}$ then by Lemma 4.58 $\overline{\mathbf{H}}$ is a c-metric space with skeleton $\mathbf{H}$, and so it fulfills 1). Further, from Corollaries 4.37 and 4.32 we have that 2) and 3) hold, as well. Notice only that when $\operatorname{diam}(\mathbf{H}) = 2$ then there exists precisely one maximal (2)-set in $\overline{\mathbf{H}}$, namely the whole $V(\mathbf{H})$.

On the other hand, the other implication follows from Lemma 4.59 for $\operatorname{diam}(\mathbf{H}) \leqslant 1$ and from Theorems 4.47 and 4.49 for $\operatorname{diam}(\mathbf{H})$ being 2 and 3, respectively (bearing in mind that in the latter case the metric with which $\overline{\mathbf{H}}$'s skeleton is equipped is in fact $d_{\mathbf{H}}$). □

5 Open problems

As a conclusion to this thesis, we propose a few open problems which are definitely worthy a considering.

1. Notice how for any polymorphism-homogeneous c-metric space $\mathcal{M}$ with a skeleton of diameter 3 the first three condition of Theorem 4.49 are satisfied. From Lemma 2.7 (for $k = 1$) follows the homomorphism-homogeneity of $\mathcal{M}$, which together with Proposition 4.15, Lemma 4.37 and Corollary 4.32 confirms our claim. What remains to be resolved still is whether there has to exist only one single 2-distance in such an $\mathcal{M}$.

2. With a view to completing the classification of countable polymorphism-homogeneous $n\star$-metric spaces, one would need to further investigate c-metric spaces with skeletons of diameter $\geqslant 4$.

3. Upon solving 2., the next step would naturally be to examine infinite metric spaces (both with and without the $\star$-property) with regard to polymorphism-homogeneity.

All in all, the search for all the polymorphism-homogeneous metric spaces proceeds, whereas this thesis, with its humble contribution, certainly provides a fairly good starting point.

6 References

[1] K. A. Baker and A. F. Pixley. Polynomial interpolation and the Chinese remainder theorem for algebraic systems. Math. Z. **143**, 165–174 (1975)

[2] P. J. Cameron. The Rado graph and the Urysohn space. Pre-print (2015)

[3] P. J. Cameron. The random graph. In The Mathematics of Paul Erdös (J. Nešetřil and R. L. Graham, eds), Springer, Berlin, pp. 331–351 (1996)

[4] P. J. Cameron and J. Nešetřil. Homomorphism-homogeneous relational structures. Combin. Probab. Comput. **15**, 91–103 (2006)

[5] D. M. Clark and B. A. Davey. Natural Dualities for the Working Algebraist. Cambridge Studies in Advanced Mathematics, vol. **57**. Cambridge University Press, Cambridge (1998)

[6] C. Delhommé, C. Laflamme, M. Pouzet, N. Sauer. Divisibility of countable metric spaces. European Journal of Combinatorics **28(6)**, 1746–1769 (2007)

[7] I. Dolinka. The Bergman property for endomorphism monoids of some Fraïssélimits. Forum Math. **26**, 357–376 (2014)

[8] Z. Farkasová, D. Jakubíková-Studenovská. Polymorphism-homogeneous monounary algebras, Math. Slovaca **65**, No. 2, DOI: 10.1515/ms-2015-0028, 359–370 (2015)

[9] T. Feller. Klassifikation der endlichen polymorphismushomogenen Turniere mit Schleifen, Masterarbeit, Technische Universität Dresden (2018)

[10] W. Hodges. Model theory. Cambridge University Press, Cambridge (2008)

[11] A. Ilić, D. Mašulović, U. Rajković. Homomorphism-homogeneous graphs with loops. Novi Sad J. Math., vol. **42**, No. 1, 59–66, (2012)

[12] K. Kaarli. Compatible function extension property. Algebra Universalis **17**, 200–207 (1983)

[13] D. Mašulović. Classifying homomorphism-homogeneous structures, WHS 2012, Prague, (2012)

[14] D. Mašulović, R. Nenadov, N. Škorić. Finite irreflexive homomorphism-homogeneous binary relational systems. Novi Sad J. Math. Vol 40, No. 3 (2010), 83–87.

[15] C. Pech and M. Pech. On polymorphism-homogeneous relational structures and their clones. Algebra Universalis, 73 (2015), 53–85.

[16] M. Pech. Local methods for relational structures and their weak Krasner algebras. Ph.D. thesis, University of Novi Sad (2009)

[17] R. Rado. Universal graphs and universal functions. Acta Arith. **9** 331–340 (1964)

[18] B. A. Romov. Extendable local partial clones. Discrete Math. **308**, 3744–3760 (2008)

[19] B. A. Romov. On the extension of not everywhere defined functions of many-valued logic. Cybernetics **23**, 319–327 (1987)

[20] B. A. Romov. Positive primitive structures. J. Mult.-Valued Logic Soft Comput. **17**, 581–589 (2011)

[21] M. Rusinov and P. Schweitzer. Homomorphism-Homogeneous Graphs. Journal of Graph Theory, DOI 10.1002/jgt.20478 (2010)

[22] M. Ó. Searcóid. Metric Spaces. Springer-Verlag London Limited (2007)

7 Curriculum Vitae

I was born on April, 25^{th} 1995 in Novi Sad. After first 6 years of primary school education I have spent the remaining two at grammar school "Jovan Jovanović Zmaj," in a special class for the gifted students. The following 4 years of my secondary education I remained at the very same school being enrolled in a special mathematical profile. Throughout all those years I have proven myself to be a distinguished contestant of a number of competitions in mathematics, physics, programming as well as in the English language, on varying levels. I have won two bronze medals at the mathematics country competitions in 2007 and 2008. In the year of 2014 I was bestowed the "Jovan Jovanović Zmaj diploma" and the "Vuk Karadžić diploma" in recognition of my exceptional work and achievements. The same year I entered the studies of pure mathematics at the University of Novi Sad, Faculty of Sciences. I have successfully completed my Bachelor's degree in 2017 with the average mark 10.0, and have continued my postgraduate studies, having passed all the exams with top marks. Due to my continual outstanding results I have been financially supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia multiple times in the form of 1-year-long scholarships. Additionally, I have received special recognitions by my University, having been issued with numerous exceptional awards for the overall outstanding success throughout my studies, together with the exceptional award for the best scientific student paper for the school year of 2016/17. What is more, I have also been awarded the prestigious "Dositeja" Scholarship for talented students issued by the Ministry of Youth and Sports of the Republic of Serbia twice, for the school years 2016/17 and 2018/19. I attended a number of scientific conferences at home and abroad, were I have given one talk so far with quite a few more to come. My scientific interests lie primarily in the fields of discrete mathematics and model theory, whereas I am particularly fond of homogeneous structures as well as the Ramsey theory. Furthermore I am the coauthor of four scientific papers.

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Pojam homomorfizam-homogenosti je prvi put uveden od strane P.J.Cameron-a i J.Nesetril-a u 2002, kao oslabljena verzija homogenosti. Struktura je homomorfizam-homogena ukoliko se svaki homomorfizam između konačno generisanih podstruktura može proširiti do endomorfizma strukture. Još jedna generalizacija, u vidu polimorfizam-homogenosti, je usledila 2014 od strane C.Pech-a i M.Pech. Za strukture sa ovom osobinom važi da se svaki lokalni polimorfizam može proširiti do globalnog polimorfizma strukture u pitanju. Cela teza se bavi istraživanjem homomorfizam-homogenih i polimorfizam-homogenih metričkih prostora.

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Abstract: **AB**	The notion of homomorphism-homogeneous structures was first brought to light by P. J. Cameron and J. Nesetril in 2002, as a relaxed version of homogeneity. We say that a structure is homomorphism-homogeneous if every homomorphism between finitely generated substructures extends to an endomorphism of the structure. Another generalization in the polymorphism-homogeneity, ensued in 2014 by C. Pech and M. Pech, with structures possessing this property in case any local polymorphism may be extended to a global polymorphism of the structure in question. The whole thesis revolves around exploring the homomorphism-homogeneous and polymorphism-homogeneous metric spaces.
Accepted by the Scientific Board: **ASB**	
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Thesis defend board: **DB**	
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