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Pogled na parcijalne diferencijalne jednačine

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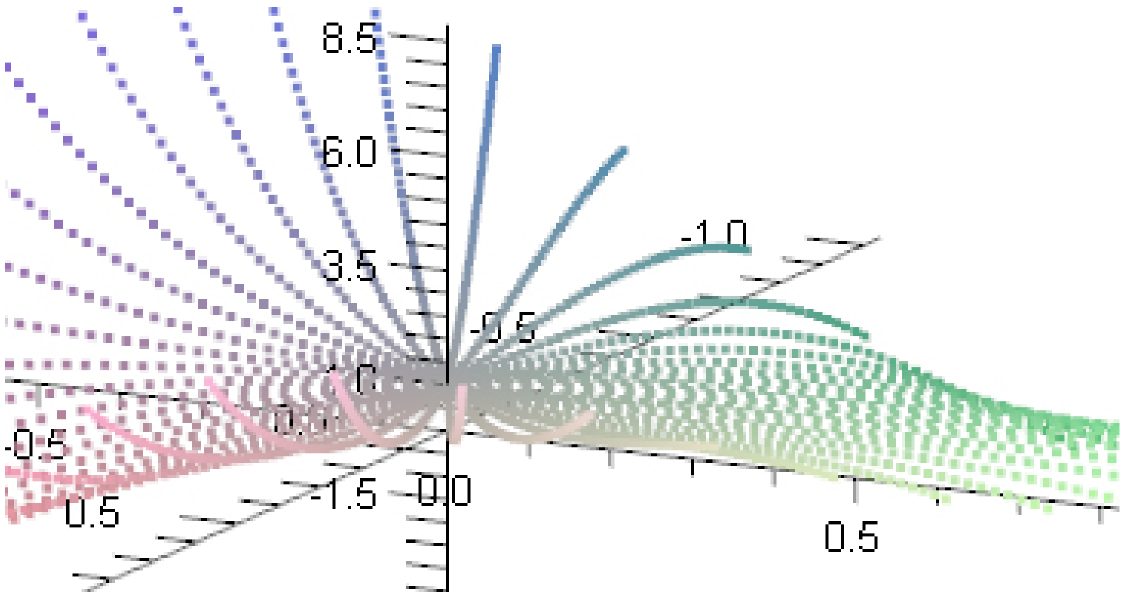
1. PARCIJALNE DIFERENCIJALNE JEDNAČINE

1.1. UVOD

PDJ je kraće ime za parcijalne diferencijalne jednačine, koje imaju jako mnogo upotreba u različitim poljima aktivnosti. To je tip diferncijalnih jednačina, uključuje nepoznatu funkciju sa nekoliko nezavisnih varijabli i njihove parcijalne izvode u odnosu na ove varijable.



SLIKA 0.1 PRIMER PDJ

Kako bi formulisali i našli rešenje za različite probleme koji uključuju funkcije nekoliko varijabli, možemo koristiti PDJ. Problemi su različiti, od zvučnih do toplotnih, elektrostatičnih, elektrodinamičnih, fluidnih, elastičnih pojava i najbolje opisuju u kojem polju parcijalne diferencijalne jednačine mogu biti korišćene.

Da istaknemo " različiti fizički fenomeni, pojave, mogu biti identično matematički formulisani i stoga vođeni samom dinamikom koja stoji iza njih". Multidimenzionalni problemi se opisuju sa PDJ, a sa druge strane one su standardne diferencijalne jednačine modeli su dinamičke sisteme.

Parcijalna diferncijalna jednačina za funkciju $u(x_1,...x_n)$ je u formi:

$$F(x_1, \cdots x_n, u, \frac{\partial}{\partial x_1}u, \cdots \frac{\partial}{\partial x_n}u, \frac{\partial^2}{\partial x_1 \partial x_1}u, \frac{\partial^2}{\partial x_1 \partial x_2}u, \cdots) = 0$$

Ako F predstavlja linearnu funkciju od u, zamenom u sa v+w se može napisati kao:

$$F(v) + F(w)$$

Sa druge strane, ako zamenimo u sa *ku*, onda je F:

$$k \cdot F(u)$$

U slučaju kada F predstavlja linearnu funkciju od u i izvodi stoga PDE su isuviše linearni. Na primer, postoji sličan slučaj parcijalnih lienarnih jednačina koji uključuje toplotne jednačine, talasne jednačine, Laplace jednačine. Sve ce biti kasnije pojašnjene.

Prvo da vidimo jednostavnu formu parcijalne diferencijalne jednačine:

$$\frac{\partial}{\partial x}u(x,y)=0\,.$$

Ako je dobro pogledamo, može se zaključiti da je funkcija u(x,y) nezavisna od x. To znači da je generalno rešenje imlicirano relacijom:

$$u(x,y)=f(y),$$

Tako da za poznato f u predstavlja arbitarnu funkciju od y i analog za standardne diferencijalne jednačine kao sto je ispod:

$$\frac{du(x)}{dx}=0$$

Sledeće rešenje za ovo je:

$$u(x)=c,$$

Ovde c predstavlja konstantnu vrednost nezavisnu od vrednosti x.

Stoga, generalno rešenje parcijalne diferencijalne jednačine uključuje arbitarnu funkciju i kao rešenje toga nije jedinstvena moramo odrediti granicu ili relaciju regiona gde je rešenje specijalno definisano.

1.2. POREKLO I KARAKTERISTIKE

Upotreba Picard-Lindelofove teoreme nam može dati veoma pogodno i produktivno rešenje za tip diferncijalne jednačine u pur interesovanju. Ove jednačine su veoma specijalne ili jedinstvene tako da one imaju specijalno poreklo ili postojanje. Druga teorema je generalna, Cauchy-Kowalevski i "stoji da Cauchi problem za bilo koju parcijalnu diferencijalnu jednačinu koja je analitična, nepoznata funkcija i njeno izvodjenje ima jedinstveno analitičko rešenje".

Patologičko ponašanje kao posledica Cauchy problema može biti prezentovano u primeru ispod i to zavisi od vrednosti n:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

Po Laplace-ovoj jednačini granični uslovi će ubuduće odrediti to kao:

$$u(x,0)=0,$$
$$\frac{\partial u}{\partial y}(x,0)=\frac{\sin nx}{n},$$

Ovde je n integralna vrednost, u i kao izvod od u, teži 0 kada n raste. Rešenje za ovo s obzirom na y biva:

$$u(x,y)=\frac{(\sinh ny)(\sin nx)}{n^2}.$$

Ovo rešenje pristupa beskonačnosti ako *nx* nije integralna višestruka vrednost od π za bilo koju vrednost koja nije nula za y. Cauchy-jev problem za Laplace jednačinu je dakle loše postavljen (ill-posed), s obzirom da rešenje ne zavisi neprekidno od podataka problema. Ovako postavljen problem nije obicno zadovoljavajući za fizičke aplikacije.

1.3. OBELEŽAVANJA

Parcijalni izvodi se često označavaju u parcijalnim diferencijalnim jednačinama indeksima na sledeći način:

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

Za prostorne izvode često je korišćen delta a za vremenske izvode se koristi dot. Što se tiče delte, može biti zamenjen Cartezijevim koordinatama :

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

Lep primer za ove dve vrednosti i njihove upotrebe bi bila talasna jednačina, često korišćena i prezentovana u dve forme:

$$\ddot{u} = c^2 \nabla^2 u$$

$$\ddot{u} = c^2 \Delta u$$

Prva je za fizičko obeležavanje, a druga za matematičko. Vrednost Δ je Laplace operator i moramo biti pažljivi da to ne pomešamo sa delta operatorom zato što imaju istu oznaku. Stoga, ovo bi bilo otprilike sve o obeležavanju kako bi bili dobro upoznati i pažljivo to koristili.

Da vidimo neke primere kako bi bolje razumeli sve navedeno.

Na primer, tu je jednačina koja objašnjava kondukciju toplote za homogeno telo i data je u dimenziji poput ove dole, jednoj dimenziji:

$$u_t = \alpha u_{xx}$$

Tako da tu je temperatura predstavljena sa u(t,x) a sto se tiče α, predstavlja pozitivnu konstantu za udeo difuzije. Nadalje mora biti specijalizovana vrednost od $u(0,x) = f(x)$ a za arbitarnu funkciju imamo f(x). Ovo je nazvano Cauchijevim problemom i ima neka genralna rešenja kao što je navedeno.

Može biti rešeno korišćenjem odvajanje od varijabli metode i korišćenjem redova u vezi toplotnih jednačina. Za periodične serije u slučaju od f a što se tiče ne periodičnih transfromacija imamo Fourier transformacije. Na poslednjem primeru, korišćenjem Fourier transformacije, rešenje može biti dato kao što je dole i to je rešenje toplotne jednačine:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-\alpha \xi^2 t} e^{i\xi x} d\xi,$$

Ovde F predstavlja arbitarnu funkciju I Fourierovu transformaciju od f za podršku inicijalnim uslovima:

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx.$$

Izvor toplote je ovde predstavljen kao f i dovljno je malen, ali dovoljno i snažan izvor, tako da će dalje biti integrisan i ići kroz delta distribuciju. Ovaj način će uključiti jačinu izvora i u slučaju kada je izvor dobro normalizovan za vrednost od 1, imamo rezultat:

$$F(\xi) = \frac{1}{\sqrt{2\pi}},$$

Konačno, izračunat je rezultat toplotne jednačine i nazvan Gausianov integral.

$$u(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\alpha\xi^2 t} e^{i\xi x} d\xi.$$

Drugi način da se ovo predstavi je:

$$u(t,x) = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

Ovde x ide ujedno sa normalnom verovatnoćom i ima vrednost o kao i varijansa 2αt koja je i ova jednačina je konačna, može biti korišćena u mnogim istraživanjima uzevši u obzir difuzione jednačine u različitim slučajevima ili pojavama.

1.4. PROSTORNA DIMENZIJA ZA TALASNU JEDNAČINU

Nepoznata funkcija za jednačinu talasne funkcije $u(t, x)$ je forma:

$$u_{tt} = c^2 u_{xx}.$$

Ovde je

- **u**- vibracija raširene žice u ravnoteži, drugi model za u je da bude varijacija u vazdušnom pritisku u cevi,
- **c**-broj koji odgovara brzini talasa.

Što se tiče *u*, takođe može biti veličina magnetnog polja u cevi. Inicijalne podaci su početni položaj žice I njena početna brzina, te tako:

$$u(0,x) = f(x),$$
$$u_t(0,x) = g(x),$$

Ovde su date arbitarne funkcije u obliku *f* i *g* tako da je finalno rešenje za problem d'Alembertova jednačina:

$$u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.$$

Ali kako bi je bolje razumeli moramo biti upoznati sa faktorima koji utiču i tako, rešenje na (t,x) je pod uticajem podataka na početnoj liniji i tako, karakteristične krive:

$$x - ct = \text{constant}, \quad x + ct = \text{constant},$$

Propagiranjem sa brzinom c, signali su odgovarajući za krivu predstavljenu iznad i stoga unazad i unapred, tako da imamo prenosiv uticaj podataka na različitim tačkama koje su date i na nekoj inicijaloj linije. Što se tiče linije, za konačna brzina c nema efekat iza trougla kroz tu tačku čije su strane karakteristične krive.

“Ovo ponašanje je veoma različito od rešenja za toplotnu jednačinu gde je efekat tačke izvora vidljiv svakoj tački u prostoru.” ([1])

U slučaju kada je t negativno, imamo validno rešenje od pre, gore, i jasno je da naše rešenje zavisi od činjenice o veoma dobro postavljenim talasnim jednačinama Cauchy problem opisuje, oba pravca, unapred i unazad.

1.5. SFERNI TALASI

Dalje će biti opisana upotreba diferencijalnih jednačina u sfernim talasima. Jedna činjenica je u vezi talasa kao što je ovaj i mora biti proračunata u tipu jednačina kao ove što su, u zavisnosti od talasa od radijalne udaljenosti f od centralne tačke izvora. Šta to znači? Pa, za ove talase je predstavljena trodimenzionalna talasna jednačina ispod, i ovde je takođe zadovoljena jedno-dimenzionalna talasna jednačina. Prvo imamo ovu formu jednačine:

$$u_{tt} = c^2 \left[u_{rr} + \frac{2}{r} u_r \right].$$

A kasnije može biti transformisana u ovo:

$$(ru)_{tt} = c^2 \left[(ru)_{rr} \right],$$

Što je ekvivalentno

$$u(t,r) = \frac{1}{r} \left[F(r - ct) + G(r + ct) \right],$$

Kao svojom finalnom i možda najrazumljivijom verzijom. Ovde su F i G arbitarne funkcije kao što se može i pretpostaviti. U slučaju kada je G jednako nuli, tada je radijacija od antene pogodna za ovaj slučaj. Ovde se može zaključiti da nema distorzije u vremenu kada je talasi oblik formiran i predstavljen od antene. U slučaju kada su predstavljene dve specijalne dimenzije, buduća iI nedistorzirana propagacija talasa nije prikazana.

1.6. LAPLACE JEDNAČINA U DVE DIMENZIJE

A nepoznatu funkciju dve varijable, data forma za φ u Laplaceovoj jednačini je sledeća:

$$\varphi_{xx} + \varphi_{yy} = 0.$$

Dobro poznata, harmonična funkcija, je rešenje za ove jednačine.

1.6.1. Veza sa holomorfnom funkcijom

Ne možemo odvojiti Laplace jednačinu i analitičku jednačinu od kompleksne i varijable zato što su usko povezane u rešenjima i stoga opisuju neki kompleks varijable te imamo realne i imaginarne delove bilo koje analitičke funkcije koja je konjugativna harmonična funkcija.

Zadovoljavajući Laplace jednačinu, možemo koristiti gradijente u ortogonalnoj formi i stoga f predstavlja vrednost od u=v. Nakon ovog stanja Cracht Riemanove jednačine je sledeće:

$$u_x = v_y, \quad v_x = -u_y,$$

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

"Konverzno, data harmonicna funkcija u dve dimenzije, je realni deo analiticke funkcije, barem lokalno." Ako tražimo detalje, tu je Laplace jednačina. ([1])

1.6.2. Problem ograničenja

Često je za Laplace jednačine i rešenje mora biti predstavljeno na način da ispuni arbitarnu vrednost na granici domena. Šta to znači? Dati primeri koji sadrže harmonične funkcije i uzimaju vrednost $u(\theta)$ na krugu radijusa u vrednosti 1.

$$\varphi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\theta')} u(\theta')d\theta'.$$

To je rešenje od Poisona i dopunjeno je kasnije nekim podacima iz Petrovskijevog istraživanja. Po Petrovskom, formula iznad može lako biti upoznata sa Fourier serijom za vrednost od φ .

U slučaju kada je r manje od 1, sledeći izvodi će lako biti izračunati u razlike ispod integralnog znaka. Ovde može biti proverena vrednost od φ koje je analitično. To je stabilno, čak i u slučaju kada je predstavnik neprekidna, ne obavezno diferencijabilna funkcija.

Rešenja kao eliptične parcijalne jednačine sadrže sledeće podatke.

" Rešenja mogu biti donešena mnogo lakše nego podaci o ograničenjima. Ovo nije kontrastno rešenjima talasnih jednačina i generalno hiperbolično parcijalno diferencijalnim jednačinama, koje tipično nemaju više izvoda nego podataka." ([1])

1.7. EULORI-TRITICOMI JEDNAČINA

U istraživanjima transoničnih tokova mozemo koristiti jednačinu pronađenu od strane Eulera i Triconija, nazvanu Eulori-Triconi jednačina:

$$u_{xx} = xu_{yy}.$$

1.8. ADVECTION JEDNAČINA

Transport konzervativnog skalara ψ u frekvenciji polja $\mathbf{u} = (u, v, w)$ je predstavljen sledećom jednačinom ili možemo reći, advektivnom jednačinom. To je:

$$\psi_t + (u\psi)_x + (v\psi)_y + (w\psi)_z = 0.$$

U slučaju kada je polje frekvencija solenoidno a to je $\nabla \cdot \mathbf{u} = 0$, tada jednačina može biti pojednostavljena u :

$$\psi_t + u\psi_x + v\psi_y + w\psi_z = 0.$$

To je jednodimenzionalni slučaj gde je u konstanta i jednaka ψ, jednačina je predstavljena kao Burgerova.

1.9. GINZBURGOVA LANDAU JEDNAČINA

Za modeliranje superkonduktiviteta koristimo ovu Ginzburg-Landau jednačinu, a to je:

$$iu_t + pu_{xx} + q|u|^2 u = i\gamma u$$

Gde su $p, q \in \mathbb{C}$ I $\gamma \in \mathbb{R}$ konstantni, a i je imaginarna jedinica.

1.10. DYM JEDNAČINA

Dym jednačina je imenovana po Hariju Dimu i javlja se u istraživanjima solitona. To je:

$$u_t = u^3 u_{xxx}.$$

1.11. VIBRIRAJUĆA NIT

Rastezanje niti medju dve tačke gde je x=0 i x=L denote amplitude zamene niti, može lako biti predstavljen za jednodimenzionalne talasne jednačine u region gde je 0<*x*<*L i t* neograničeno. Okolnosti sadrže slučaj kada je nit vezana dole na krajevima tako granični uslovi mogu biti predstavljeni na ovaj način u redu da odgovore na uslove:

$$u(t,0) = 0, \quad u(t,L) = 0,$$

Ovde su pomenuti inicijalni uslovi:

$$u(0,x) = f(x), \quad u_t(0,x) = g(x).$$

Za metod, odvajanje varijabli od talasnih varijabli može biti urađeno kroz:

$$u_{tt} = c^2 u_{xx},$$

I tako, rešenje će biti

$$u(t,x) = T(t)X(x),$$

Ovde je sa konstantnom vrednosti k i odredjene vrednosti, jednačina predstavljena kao:

$$T'' + k^2c^2T = 0, \quad X'' + k^2X = 0,$$

Granični uslovi dalje impliciraju da je x višestruko od sinkx i k mora imati oblik

$$k = \frac{n\pi}{L},$$

Ovde je n ceo broj. Tako da smo došli do ove bitne tačke. Prethodne informacije su bile potrebne kako bi znali sledeći član gde je n ceo broj. Svaki član reda odgovara jednom modu vibracie niti.

Mode sa vrednosti n=1 predstavlja fundamentlani mod.

Frekvencije reda moda su sve višestruke od ove frekvencije.

Frekvencije će formirati redove koje će izaći iz niti,i one su baza za muzičku akustiku. Sledeći korak će biti postavljanje ili dopunjavanje inicijalnih uslova i najbolji način da se to učini je upotreba prezentacije f kao g i kao konačne sume modela poput ovog.

"Duvački instrumenti tipično odgovaraju na vibracije vazdušne kolumne sa jednim otvorom i jednim krajem zatvorenim." ([1])

Tako, odgovarajauće granice bi bile:

$$X(0) = 0, \quad X'(L) = 0.$$

Primenom metoda separacije u ovakvim slučajevima kao što je prezentovani, je rešenje i prateći ga, bićemo vođeni do nekoliko čudnih prekotonova.

Formiranje rešenja problema ovog tipa možemo koristiti teoriju pronadjenu od strane Sturma i Liovila.

1.12. VIBRIRAJUĆE MEMBRANE

Širenje membrane preko krive C koja formira granicu domena D u nekim određenim ravnima, talasna jednačina pratećeg tipa može upravljati ovom, mambranom, vibracijama

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy},$$

Ako je t>0 i (x,y) je u D.

Granični uslov je *u(t,x,y)* je na C. Metod separacije varijabli vodi ka formi:

$$u(t,x,y) = T(t)v(x,y),$$

Koja zauzvrat mora biti dopunjena sa sledećim:

$$\frac{1}{c^2}T'' + k^2T = 0,$$

$$v_{xx} + v_{yy} + k^2v = 0.$$

Sledeća jednačina je nazvana Homholtz jednačina. Dopuštajući ne trivijalnom v u redu da popuni granične uslove na C, moramo odrediti konstantu k. Da vidimo: k^2 je nazvano karakteristična vrednost Laplaciana u D a povezano rešenje je karakteristično rešenje Laplaciana u D. " Sturm Liovilova teorija može biti proširena na ovu eliptičnu eigenvalue problem." (Jost, 2002)

2. KLASIFIKACIJA

Kako bi napravili neki vodič za pogodne ili odgovarajuće granične uslove, matematičari su napravili klasifikaciju među jednačinama. Korišćenjem ovakvih klasifikacija, rešenja će doći lako prilično. Postoji nekoliko redova za klasifikaciju. Na primer, za neke linearne, drugog reda parcijalne diferencijalne jednačine klasifikacija je parabolična, hiperbolična i eliptična. Druge kao što je Euler Tricomi jednačina imaju različite tipove u različitim regionima.

2.1. JEDNAČINE PRVOG REDA

2.1.1. PDJ prvog reda

Parcijalne diferncijalne jednačine prvog reda uključuju samo prve izvode nepoznate funkcije od n varijabli nazvane parcijalnim jednačinama prvog reda. Jednačina uzima oblik:

$$F(x_1, \ldots, x_n, u, u_{x1}, \ldots u_{x_n}) = 0.$$

Ovakve jednačine proizilaze iz konstrukcije karakterističnih površina za hiperbolične parcijalne diferencijalne jednačine, u računajima varijacija, u nekim geometrijskim problemima i proizilaze u jednostavnom modelu za gas dinamiku čija su rešenja uključena u metod karakteristika.

Generalno rešenje integrisane familije uobičajenih jednačina moze biti orjentisano na rešenje iz pojedine jednačine prvog reda koja će ubuduće pomoći pronalaženju rešenja za ove pojedinačne jednačine. Tako, možemo reći da ovo generalno rešenje je obuhvaćeno u različite uobičajene diferencijalne jednačine.

2.1.2. Karakteristične površine za talasne jednačine

Karakteristične površine za talasne jednačine su nivo površina za rešenja jednačine:

$$u_t^2 = c^2 \left(u_x^2 + u_y^2 + u_z^2 \right).$$

U slučaju kada je talas postavljen kao $u_t = 1$, tu je određeni gubitak generalnih i sledećih prezentovan kao:

$$u_x^2 + u_y^2 + u_z^2 = \frac{1}{c^2}.$$

U vektorskoj rotaciji, sledi:

$$\vec{x} = (x, y, z) \quad \text{and} \quad \vec{p} = (u_x, u_y, u_z).$$

Set rešenja sa ravnima kao nivoima površina je dat kao:

$$u(\vec{x}) = \vec{p} \cdot (\vec{x} - \vec{x_0}),$$

Gde:

$$|\vec{p}| = \frac{1}{c}, \quad \text{and} \quad \vec{x_0} \quad \text{is arbitrary.}$$

Ako je x i xo sadržano fiksno, razvoj ovih rešenja je sadržan traženjem tačke na sferi radijusa 1/c gde je vrednost od u stacionarna. Ovo je istina ako je $\vec{p}$ paralelno sa $\vec{x} - \vec{x_0}$. Stoga ovaj je predstavljen jednačinom:

$$u(\vec{x}) = \pm\frac{1}{c}|\vec{x} - \vec{x_0}|.$$

Sfere sa radiusom koji proizilazi i koji imaju odskok sa frekvencijom c ne mogu ovo uzeti za odgovarajuće rešenje. Ovo su laki konusi u prostoru i vremenu.

Što se tiče problema, glavni razlog ovde bi bio odredjivanje nivoa površine S gde je u=0 za t=0. Ako uzmemo razvoj svih sfera sa centrima u S, tada ćemo uzeti u obzir i ovo rešenje, a što se tiče S, radius bi rastao sa frekvencijom koja je obeležena kao c. Ovaj zavoj je sadržan u

$$\frac{1}{c}|\vec{x} - \vec{x_0}| \quad \text{is stationary for} \quad \vec{x_0} \in S.$$

Ako postavimo ovaj deo $|\vec{x} - \vec{x_0}|$ u relaciju normalno sa S, zahtevi će biti popunjeni. Stoga zavoj odgovara kretanju sa frekvencijom c duž svake normale ka S. Ovo je Huygenova konstrukcija talasnog fronta tako da će svaka tačka na S emitovati sferni talas na vreme t-0. Onda talas će suprotstaviti na sledećem vremenu t je zavoj ovih sferičnih talasa. Normale ka S su laki zraci.

2.2. DVODIMENZIONALNA TEORIJA

Generalno diferencijalne jednačine prvog reda imaju formu:

$$F(x, y, u, p, q) = 0,$$

Gde:

$$p = u_x, \quad q = u_y.$$

Kompletan integral ove jednačine je rešenje φ(*x*,*y*,*u*) koje zavisi od dva parametra a i b.

Što se tiče parametra n, biće zahtevan u n-dimenzionalnom slučaju a ako želimo da razvijemo glatko rešenje, birajući arbitrarnu funkciju w u ovom slučaju. Kasnije će biti postavljen b, kao b=w(a). Sledeće, mi moramo da utvrdimo A:

-A(x,y,u) zahtevajući totalne izvode:

$$\frac{d\varphi}{da} = \varphi_a(x, y, u, A, w(A)) + w'(A)\varphi_b(x, y, u, A, w(A)) = 0.$$

U ovom slučaju, rešenje uw je dato kao:

$$u_w = \phi(x, y, u, A, w(A))$$

Ako pronađemo rešenje za funkciju w, biće lakše da nađemo rešenje za našu parcijalnu diferencijalnu jednačinu. Drugi način je da vodimo konstrukciju lakog konusa kao karakterističnu površinu za talasnu jednačinu.’

U slučaju kada kompletan integral nije dostupan, rešenje može još uvek biti sadržano rešavanjem sistema uobičajenih jednačina. Kako bi obuhvatili ovaj sistem, prvo zabeležiti da parcijalne diferencijalne jednačine određuju konus analogno lakom konusu na svakoj tački:

-ako je PDE linearno u izvodima od u (to je kvazi linearno) onda

-konus degeneriše u liniji.

U generalnom slučaju, parovi (p,q) koji zadovoljavaju jednačinu određuju familiju ravni na datoj tački:

$$u - u_0 = p(x - x_0) + q(y - y_0),$$

Gde

$$F(x_0, y_0, u_0, p, q) = 0.$$

Zavoj ovih ravni je konus ili linija ako je PDE quazi linearno.

Tako, što se tiče zavoja, uslovi bi bili:

$$F_p\, dp + F_q\, dq = 0,$$

F-vrednovano kao (x_0, y_0, u_0, p, q),

-dp i dq- inkrementi od p i q koji zadovoljavaju F=0.

Konačno, generator konusa je linija sa sledećim redom:

$$dx : dy : du = F_p : F_q : (pF_p + qF_q).$$

Pravac ovog generatora odgovara lakim zracima za talasnu jednačinu. Da bi integrisali diferencijalne jednačine među ove pravce, moramo da nađemo inkremente za p i q među zracima. To moze biti dalje sadržano diferenciranjem PDE:

$$F_x + F_u p + F_p p_x + F_q p_y = 0,$$
$$F_y + F_u q + F_p q_x + F_q q_y = 0,$$

Sledeće, zračni pravci u (x,y,u,p,q) prostoru bi bili sledeći:

$$dx : dy : du : dp : dq = F_p : F_q : (pF_p + qF_q) : (-F_x - F_u p) : (-F_y - F_u q).$$

2.2.1. **Jednačine drugog reda**

Ako krenemo od ove relacije $u_{xy} = u_{yx}$, generalno drugog reda parcijalnih diferencijalnih jednačina u dve odvojene varijable imaju sledeću formu:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots = 0,$$

Ovde su koeficijenti ABC zavisni od x i y. Ako je $A^2 + B^2 + C^2 > 0$ preko regiona od xy ravni, parcijalne diferencijalne jednačine su drugog reda u tom regionu.

Neke ili analogne ovoj jednačini su date ispod relacijom:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

To je dato za konusnu sekciju.

Zamenom za druge varijable kao što je učinjeno u Fourier transformaciji, ubuduće će konvertovati konstantno koeficijent i parcijalno diferencijalnu jednačinu PDE u polinominalnu istog stepena sa top nivoom ili homogenim polinominalnim, gde je kvadratna forma ili ubuduće, posebno će biti značajno za našu klasifikaciju.

"Samo kao jedna klasifikuje konusnu sekciju i kvadratnu formu u paraboličnu,hiperboličnu, eliptičnu zasnovano na diskriminanti $(2B)^2 - 4AC$, isto moze biti učinjeno za PDE drugog reda na datoj tački. Bilo kako bilo, diskriminant u PDE je dat putem $B^2 - AC$, kroz konvenciju od xy postajući 2B pre nego B; formalno diskriminant asocirane kvadratne forme je $(2B)^2 - 4AC = 4(B^2 - AC)$, što je faktor od 4 ostavljen jednostavnosti."

1. $B^2 - AC < 0$ rešenja elliptic PDEs su glatka koliko to koeficijenti dozvoljavaju unutar oblasti gde postoji rešenje. Na primer, rešenja Laplasove jednašine su analizička, čak I ako granični uslovi nisu glatki. Kretanje fluida na subsoničnim brzinama se može aproksimirati eliptičnom PDJ, Euler–Tricomi jednačina je eliptična za $x<0$.
2. $B^2 - AC = 0$: **parabolične** jednačine u svakoj tački mogu da se transformišu u oblik sličan toplotnoj jednačini smenom promenljivih. Rešenja su glatka posle početnog trenutka. Euler–Tricomi jednačina je parabolična na pravoj $x=0$.
3. $B^2 - AC > 0$: **hiperbolične** jednačine prenose diskontinuitete nastale u početnom trenutku. Primer je talasna jednačina. Kretanje fluida na supersoničnim brzinama se može aproksimirati eliptičnom PDJ, Euler–Tricomi jednačina je hiperbolična za $x>0$.

Sledeće, u slučaju da imamo nezavisne varijable za proizvoljno n:

- $x_1, x_2, \ldots, x_n$,

Onda generalna linerana parcijalna diferencijalna jednačina drugog reda ima sledeću formu:

$$Lu = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{plus lower order terms} = 0.$$

"Klasifikacija zavisi od potpisa 'katakterističnih korena koeficijenta matice."

1. Eliptična: Svi karakteristični koreni su pozitivni ili negativni.
2. Parabolična: Svi karakteristični koreni su pozitivni Ili negativni sem jednog koji je nula.
3. Hiperbolična: Samo jedan karakteristični koren je negativan, dok su ostali pozitivni ili obrnuto.
4. Ultrahiperbolična: Ima više od jednog karakterističnog korena svakog znaka i nema nula. ([2])

2.2.2. Sistemi prvog reda jednačina i karakterističnih površina

Za klasifikaciju sistema prvog reda koristimo matričnu notaciju. Nepoznata u je sada vektor sa m komponentama i koeficijenti su *m*x*m* matrice A za

$$\nu = 1, \ldots, n$$

Parcijalne diferencijalne jednačine će kasnije uzeti ovu formu:

$$Lu = \sum_{\nu=1}^{n} A_\nu \frac{\partial u}{\partial x_\nu} + B = 0,$$

Gde:

- Koeficijent matrice Ay,
- Vektor B može zavisiti od x i u.
- Ako je hiperpovršina S data u implicitnoj formi:

$$\varphi(x_1, x_2, \ldots, x_n) = 0,$$

Gde

- φ nema nula gradijent
- -onda S je karakteristična površina za operator L na datoj tački ako je karakteristična forma nestaje

$$Q\left(\frac{\partial \varphi}{\partial x_1}, \ldots, \frac{\partial \varphi}{\partial x_n}\right) = \det\left[\sum_{\nu=1}^{n} A_\nu \frac{\partial \varphi}{\partial x_\nu}\right] = 0.$$

“Geometrijska interpretacija ovih uslova je sledeća: ako su podaci za u prepisani na površinu S, onda može biti moguće da se odredi dormalni izvod u na S iz diferencijalne jednačine. Ako je podatak na S i diferencijalna jednačina determiniše normalan izvod od u na S onda S je ne karakteristična. Ako je podatak na S i diferencijalna jednačina ne određuje normalne izvode od u na S, onda je površina karakteristična.. I diferencijalna jednačina ograničava podatke na S: diferencijalna jednačina je internal za S.”

1. Sistem prvog reda $Lu=0$ je eliptičan ako neka karakteristične površi za L: Vrednosti u I njenih izvoda uvek određuju normalni izvod od u na S.
2. Sistem prvog reda je hiperboličan u nekoj tački ako postoji space-like karakteristična površ S u toj tački sa normalom ξ. To znači da za svaki netrivijalni vektor η normalan na ξ, I skalar λ, jednačina $Q(\lambda\xi + \eta) = 0$, ima realni koren $\lambda_1, \lambda_2, ..., \lambda_m$. Koreni su različiti i što se tiče sistema, hiperboličan je.

Geometrijska interpretacija ovih uslova je sledeća: " karakteristična forma $Q(\zeta)=0$ definiše konus normani konus, sa homogenim koordinatama . ζ U hiperboličnom slučaju, ovaj konus ima listu m i osa prolazi u okviru ovih listova ζ = λ ξ, a ne preseca nijednu od njih."

Razmešten od porekla putem η, osa će seći svaki list. U eliptičnom slučaju, normalni konus nema realne listove.

2.2.3. Jednačine mešovitog oblika

Parcijalne diferencijalne jednačine imaju nestabilan koeficijent i zato ne mogu biti deo bilo koje druge kategorije pre nego ove mešovitog tipa. Drugim rečima, ovaj naročiti koeficijent naših parcijalnih jednačina će biti uključen u mešoviti tip jednačine.

Ovde je dat jednostavan primer Euler tricomi jednačine nazvane eliptična hiperbolična jer je eliptična u region x<0, hiperbolična u region x>o stoga

$$u_{xx} = xu_{yy}$$

To degeneriše parabolično u liniji x=0.

2.2.4. Infinitivne PDE u kvantnoj mehanici

Quantum Hamilton jednačine za trajektore kvantumskih česticama vođeno je od strane Weyl kvantizacije u fazi prostora. Jednačine kao što su ove beskonačnog reda su parcijalne diferencijalne. U semiklasičnim ekspanzijama jedna ima beskonačan sistem od ODEs na bilo kome ustanovljenom redu od $\hbar$. Jednačina evolucije od Wingera funkcije je PDE beskonačnog reda takođe. Kvantum trajekt je kvantum karakterističan sa upotrebom gde jedan kalkuliše evolucijom Wignerove funkcije.

3. METODE REŠAVANJA I ANALIZE PDJ

3.1. INTEGRALNA TRANSFROMACIJA

Parcijalna diferencijalna jednačina može biti transfromisana u jednostavniju formu (ili recimo u formu sa razdvojenim promenljivima) od parciajlne diferencijalne jednačine. To bi bilo u skladu sa dijagonalizovanjem operatora kao sto i jeste.

Primer za ovaj način je Fourier analiza. Dijagonalizuje toplotnu jednačinu koristeći bazu karakterističnih vektora sinusoidnog talasa.

“Ako je domen konačan ili periodičan, beskonačna suma rešenja kao što su Fourierove serije ne odgovarajuća, ali integral rešenja kao što je Fourier je generalno traženo za beskonačne domene. Rešenje za tačku izvora za toplotnu jednačinu dato iznad je primer upotrebe Fourier integrala.” ([2]).

3.2. PROMENA VARIJABLI

Redukovanjem parcijalnih diferencijalnih jednačina u jednoj jednostavnijoj formi je urađeno kroz rešenje adaptne varijable promena. Ovde je dat lep primer promene ovih varijabli u jednačini pronađenoj od strane Black i Schole kao što možemo videti dole:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Može se redukovati do jednačine za toplotu:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$V(S,t) = Kv(x,\tau)$$

$$x = \ln(S/K)$$

$$\tau = \frac{1}{2}\sigma^2(T-t)$$

$$v(x,\tau) = \exp(-\alpha x - \beta\tau)u(x,\tau).$$

3.3. LIE GRUPNA METODA

Druga teorija diferencijalnih jednačina je pronađena od strane Sophusa L godine 1870. To je bilo zadovoljavajuća forma jednačine za pronalaženje rešenja. “ Pokazao je da integracija teorija starijih matematičara može, putem upoznavanja što je sada nazvano Lie grupa, biti referisano kao zajednički izvor; i taj uobičajeni diferencijalne jednačine koje priznaju istu beskonačnu transformaciju predstavljaju uporedljive poteškoće integracije”. Druga stvar je urađena, uvećavanjem subjekta transformacije sadržaja.

Karakteristike simetrije diferncijalnih jednačina korišćene su za druge načine rešavanja, sa konstantnom beskonačnošću transformacija rešenja ka rešenjima od strane Lie teorije.

Po teoriji CO grupe, da bi razumeli strukturu linearnih i nelinaernih parcijalnih diferencijalnih jednačina, korišćeni su Lie algebra i diferencijalna geometrija, a takođe u svrhu za generisanje jednačina kao i za pronalaženje:

- Lax parova, ponavljanje operatora,
- Backlund transformacija i konačno
- za nalaženje tačnih analitičkih rešenja za parcijalne diferencijalne jednačine.

U proučavanju istraživanja diferencijalnih jednačina mnoge simetrične metode su korišćene, u različitim poljima aktivnosti, na primer u matematici, fizici, inženjeringu i tako i u drugim disciplinama.

3.4. NUMERIČKE METODE ZA REŠAVANJE PDES

Da bi rešili parcijalnu diferencijalnu jednačinu, najčešće su u upotrebi tri numerička metoda:

1. Konačna element metoda (FEM)
2. Konačna vrednost metoda (FVM)
3. Konačna razlika metoda (FDM).

“FEM ima prominentnu poziciju između metoda i naročito njenih izuzetnih efikasnih viših redova verzija hp-FEM”.

Druga verzija od FEM uključuje:

- Generalizovan konačan metod
- Raširen konačan metod
- Spektralni konačan metod
- Mreža-slobodan konačan metod
- Diskontinuosni Galerkin konačan metod

3.5. METOD KONAČNIH ELEMENATA

Numeričke tehnike za nalaženje odgovarajućeg rešenja za parcijalne diferencijalne jednačine i integralne jednačine u konačnom metodu, poznatom kao FEA. To je praktična aplikacija, primena, često poznata kao konačan element analiza ili FEa kao što je već rečeno. Nalaženje rešenja je u smeru eliminisanja diferencijalnih jednačina, siguran položaj problema ili putem renderingovih parcijalnih diferencijalnih jednačina u maksimalizovanje sistema uobičajenih diferencijalnih jednačina. Ove jednačine će nadalje biti numerički integrisane korišćenjem standardnih tehnika. Na primer tu su neke tehnike kao što je Euler metoda, Runge Kutta itd.

3.5.1. Metod konačnih razlika

Za maksimalizovanje rešenja diferencijalne jednačine korišćenjem konačne razlike jednačina za izvode koriste se konačne metode razlike.

3.5.2. Metod konačnog volumena

Ukazuje na mali volume okružujući svaku čvornu tačku na mrežici. Slično konačnom metodu razlika možemo proračunati vrednosti u diskretnim mestima na mrežici geometrije. U metodu konačnog volumena, volume integral u parcijalnoj diferencijalnoj jednačini koji sadrži razliku termina je konvertovan na površinu integral, koristeći divergenciju teoreme; ovi termini su zatim izračunati kao fluksovi na površini svakog konačnog volumena.” Ove metode su konzervativne zbog fluksa koji ide u volume dat i identičan onom prilagođenom volumenu.

4. PRIMERI BITNIH PARCIJALNIH DIFERENCIJALNIH JEDNAČINA KOJE PROIZILAZE U PROBLEM MATEMATIČKE FIZIKE

Benjamin-Bona-Mahony equation

$$u_1 + u_x + u\, u_x - u_{x\,x\,1} = 0$$

Biharmonic equation

$$\nabla^4 \varphi = 0$$

Boussinesq equation

$$u_{tt} + \alpha\, u_{xx} + \beta (u^2)_{xx} + \gamma\, u_{xxxx} = 0$$

Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Chaplygin's equation

$$u_{x\,x} + \frac{y^2}{1 - \frac{y^2}{c^2}}\, u_{y\,y} + y\, u_y = 0.$$

Euler-Darboux equation

$$u_{xy} + \frac{\alpha u_x - \beta u_y}{x - y} = 0.$$

Heat conduction equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0, \\ \frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V}\mathbf{V} + p\mathbf{I}) &= 0, \\ \frac{\partial E}{\partial t} + \nabla \cdot (E + p)\mathbf{V} &= \nabla \cdot (\kappa \nabla T), \end{aligned} \quad (1)$$

Helmholtz differential equation

$$(\nabla^2 + k^2)\phi = 0$$

Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0,$$

Korteweg-de Vries-Burgers equation

$$u_t + 2 u u_x - \nu u_{xx} + \mu u_{xxx} = 0.$$

Korteweg-de Vries equation

$$u_t + u_{xxx} - 6 u u_x = 0.$$

Krichever-Novikov equation

$$\frac{u_t}{u_x} = \frac{1}{4}\,\frac{u_{x\,x\,x}}{u_x} - \frac{3}{8}\,\frac{u_{x\,x}^2}{u_x^2} + \frac{3}{2}\,\frac{p\,(u)}{u_x^2},$$

gde

$$p\,(u) = \tfrac{1}{4}\left(4\,u^3 - g_2\,u - g_3\right).$$

Laplace's equation

$$\nabla^2 \psi = 0.$$

Lin-Tsien equation

$$2\,u_{t\,x} + u_x\,u_{x\,x} - u_{y\,y} = 0.$$

Sine-Gordon equation

$$v_{t\,t} - v_{x\,x} + \sin v = 0.$$

Spherical harmonic differential equation

$$\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2} + l\,(l+1)\right] u = 0.$$

Tricomi equation

$$u_{yy} = y\, u_{xx}\,.$$

Wave equation

$$\nabla^2 \psi = \frac{1}{v^2}\,\frac{\partial^2 \psi}{\partial t^2}\,.$$

5. POJEDINAČNE PARCIJALNE DIFERENCIJALNE JEDNAČINE

a. Linear equations.

1. *Laplace's equation*

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0.$$

2. *Helmholtz's (or eigenvalue) equation*

$$-\Delta u = \lambda u.$$

3. *Linear transport equation*

$$u_t + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

4. *Liouville's equation*

$$u_t - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

5. *Heat (or diffusion) equation*

$$u_t - \Delta u = 0.$$

6. *Schrödinger's equation*

$$iu_t + \Delta u = 0.$$

7. *Kolmogorov's equation*

$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

8. *Fokker–Planck equation*

$$u_t - \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

9. *Wave equation*

$$u_{tt} - \Delta u = 0.$$

10. *Klein–Gordon equation*

$$u_{tt} - \Delta u + m^2 u = 0.$$

11. *Telegraph equation*

$$u_{tt} + 2du_t - u_{xx} = 0.$$

$$u_{tt} + 2du_t - u_{xx} = 0.$$

12. *General wave equation*

$$u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

13. *Airy's equation*

$$u_t + u_{xxx} = 0.$$

14. *Beam equation*

$$u_{tt} + u_{xxxx} = 0.$$

---

b. Nonlinear equations.

1. *Eikonal equation*

$$|Du| = 1.$$

2. *Nonlinear Poisson equation*

$$-\Delta u = f(u).$$

3. *p-Laplacian equation*

$$\operatorname{div}(|Du|^{p-2} Du) = 0.$$

4. *Minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right) = 0.$$

5. *Monge–Ampère equation*

$$\det(D^2u) = f.$$

6. *Hamilton–Jacobi equation*

$$u_t + H(Du, x) = 0.$$

7. *Scalar conservation law*

$$u_t + \operatorname{div}\mathbf{F}(u) = 0.$$

8. *Inviscid Burgers' equation*

$$u_t + uu_x = 0.$$

9. *Scalar reaction-diffusion equation*

$$u_t - \Delta u = f(u).$$

10. *Porous medium equation*

$$u_t - \Delta(u^\gamma) = 0.$$

11. *Nonlinear wave equation*

$$u_{tt} - \Delta u + f(u) = 0.$$

12. *Korteweg–deVries (KdV) equation*

$$u_t + uu_x + u_{xxx} = 0.$$

13. *Nonlinear Schrödinger equation*

$$iu_t + \Delta u = f(|u|^2)u.$$

[3]

6. SISTEMI PARCIJALNIH DIFERENCIJALNIH JEDNAČINA

Linear systems.

1. *Equilibrium equations of linear elasticity*

$$\mu\Delta\mathbf{u} + (\lambda + \mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

2. *Evolution equations of linear elasticity*

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda + \mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

3. *Maxwell's equations*

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B} \\ \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0. \end{cases}$$

Nonlinear systems.

1. *System of conservation laws*

$$\mathbf{u}_t + \operatorname{div}\mathbf{F}(\mathbf{u}) = \mathbf{0}.$$

2. *Reaction-diffusion system*

$$\mathbf{u}_t - \Delta\mathbf{u} = \mathbf{f}(\mathbf{u}).$$

3. *Euler's equations for incompressible, inviscid flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

4. *Navier–Stokes equations for incompressible, viscous flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} - \Delta\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

[3]

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Abear Saeed Aboglida

A walk through partial differential equation

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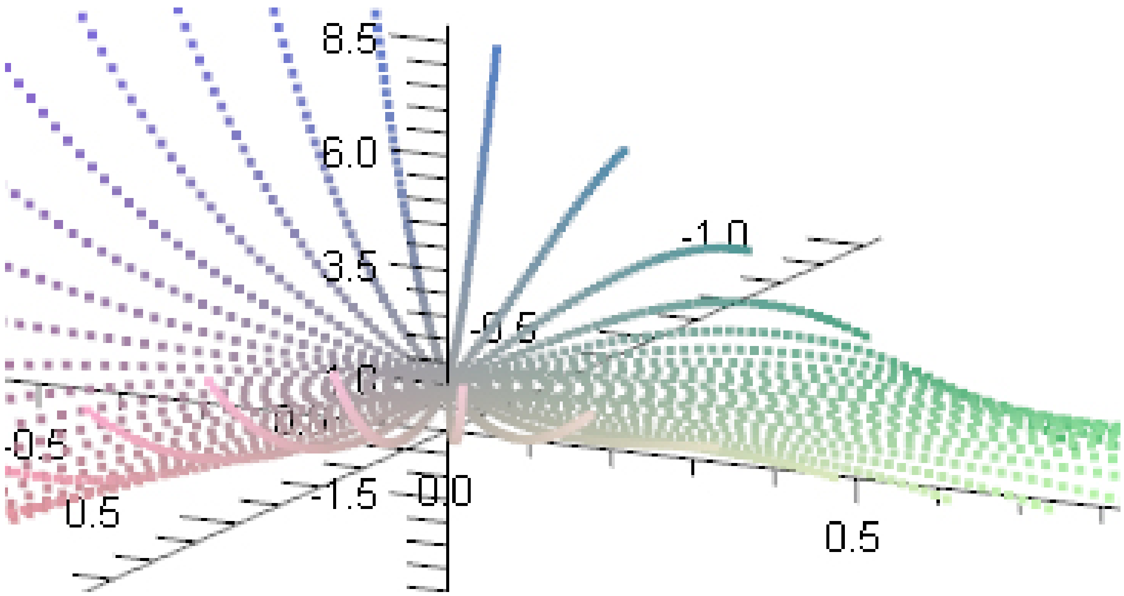
1. PARTIAL DIFFERENTIAL EQUATION

1.1. INTRODUCTION

PDE is shorter name for partial differential equations ,which have so many usages in different fields of acting. It is type of differential equation. It involves function that is unknown for several independent variables and for their partial derivates with respect to those variables.



Set of partial differential equations (example of values) [4]

In order to formulate and make solution for different problems that include functions of several variables, we can use this type of equations. Variables are different, from sound to heat, electrostatics, electrodynamics, fluid flow, elasticity and these variables best describe in which field partial deferential equations can can be used. [1]

To point up, “distinct physical phenomena may have identical mathematical formulations and thus be governed by the same underlying dynamics”. Multidimensional systems are modeld by these types of equastion, and on the other hand there are ordinary differential equations modeling dynamic systems.

A partial differential equation for the function $u(x_1,...x_n)$ is of the form:

$$F(x_1, \cdots x_n, u, \frac{\partial}{\partial x_1}u, \cdots \frac{\partial}{\partial x_n}u, \frac{\partial^2}{\partial x_1 \partial x_1}u, \frac{\partial^2}{\partial x_1 \partial x_2}u, \cdots) = 0$$

F represents a linear function of u. It derivatives by replacing u with v+w so we can write F as:

$$F(v) + F(w)$$

On the other hand, if we replace u with ku, than F is explained through this form:

$$k \cdot F(u)$$

In case when F represents linear function of u and its derivates than PDE is too linear. For example, there are common cases of linear partial differential equation that includes heat equation, wave equation, Laplace equation. All will be explained later.

First, let’s see simple form of partial differential equation:

$$\frac{\partial}{\partial x}u(x,y) = 0\,.$$

If we take a look at it, we can conclude that the function u (x,y) is independent of x. It means that general solution implies on relation:

$$u(x,y) = f(y),$$

So for f is known that represents arbitrary function of y and the analogous for ordinary differential equation is as below:

$$\frac{du(x)}{dx} = 0$$

Further solution for this is:

$$u(x) = c,$$

here c represents constant value independent from value x.

Hence, ‘general solutions of partial differential equation involve arbitrary functions’ and as solution of it isn’t unique we must specified the boundary or relation of the region where is this solution specially defined. [1]

1.2. ORIGIN AND CHARACTERISTIC

Usage of Picard-Lindelof theorem can give us very suitable and productibel solution for type of differential equation in pur interest. These equations are very special or unique so as they have special origin or existence. Another theorem is general, Cauchy-Kowalevski and “it states that the Cauchy problem for any partial differential equation that is analytic in the unknown function and its derivatives has a unique analytic solution”.

Pathological behavior as the sequence of Cauchy problems can be presented in the example below and it depends on value n:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

By Laplace equation so boundary conditions will further order it as:

$$u(x,0) = 0,$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\sin nx}{n},$$

Here, integral value is n and as for the derivative of u, there is approach O in x as n in increasing value so solution for this respect for y will be:

$$u(x,y) = \frac{(\sinh ny)(\sin nx)}{n^2}.$$

"This solution approaches infinity if *nx* is not an integer multiple of π for any non-zero value of *y*. The Cauchy problem for the Laplace equation is called *ill-posed* or *not well posed*, since the solution does not depend continuously upon the data of the problem. Such ill-posed problems are not usually satisfactory for physical applications." [1]

1.3. NOTATION

Partial derivates are often donetes in partial differential equations and also they use subscripts as below:

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right).$$

For spatial derivatives is often used del and for time derivates is used dot. As for del, it can be explained in Cartesian coordinates below:

$$\nabla = (\partial_x, \partial_y, \partial_z)$$

Nice example for these two values and their usages will be wave eqation, often used and presented down in two forms:

$$\ddot{u} = c^2 \nabla^2 u$$

$$\ddot{u} = c^2 \Delta u$$

First one is for physicis notation and second for math notation. Value Δ is Laplace operator and we must be careful not to mix up with delta operator because they have the same sign. SO, that is all about notation, to be well introduced and carefully use it.

Let's see some examples in order to better understand all of this.[1]

For example, there is eqation explaining conduction of heat for a homogenous body and it is given in dimension as below, one dimension:

$$u_t = \alpha u_{xx}$$

So here it temperature presented with u(t,x) and as for α, it represents positive constant for ratio of diffusion. Further must be specified value of $u(0,x) = f(x)$ and for arbitrary function we have f(x). This is so called the Cauchy problem and has some general solutions as below. [1]

It can be solved with usage of separation of variables method and usage of heat equation article. For periodic series in case of f and as for non-periodic transforms we have Fourier transforms. By the last example, using Fourier transform, solution can be given as below and it is solution of the heat equation:

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-\alpha \xi^2 t} e^{i\xi x} d\xi,$$

Here F represents ana arbitrary function and Fourier transform of f for supporting initial conditions:

$$F(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x}\, dx.$$

Source of heat is here represented by f and it is small but enough strong source so further will integral go through delta distribution. This way will include the strength of the source and in case when source is well normalized for value of 1, we have result as below:

$$F(\xi) = \frac{1}{\sqrt{2\pi}},$$

Finally, there is calculated result of the heat equation and it is called Gaussian integral.

$$u(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha\xi^2 t} e^{i\xi x} d\xi.$$

Another way to express it is:

$$u(t,x) = \frac{1}{2\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right).$$

Here x goes according to normal probability and has mean 0 so as variance $2\alpha t$ and this equation finaly, can be used in many researches considered by diffusion eqations in different cases or phenomena. [1]

1.4. SPATIAL DIMENSION FOR WAVE EQUATION

Unknown function for equation of the wave function $u(t, x)$ in this form

$$u_{tt} = c^2 u_{xx}.$$

Here is:

u- the displacement of stretched string from equilibrium, another solution for u is to be difference in air pressure in a tube,

c- number that corresponds to the velocity of the wave.

As for u it also can be the magnitude of an magnetic field in a tube. Prescribing the initial displacement and velocity for the Cauchy problem for these forms above are displacement and velocity of a strong or it can also be other type of medium, so:

$$u(0,x) = f(x),$$
$$u_t(0,x) = g(x),$$

Here are given arbitrary functions in forms f and g so final solution for this problem is d'Alembert's equation:

$$u(t,x) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\,dy.$$

But in order to understand it well, we must be introduced with influencing factors and so, the solution at (t,x) is influenced by the data on the segment of the initial line and so, characteristic cruve:

$$x - ct = \text{constant}, \quad x + ct = \text{constant},$$

Propagating with velocity c, signals are corresponding for curves represented above and therefore forward and backward, so we have propagating the influence of the data at different points that are given and on some initial line. As for line, the finite velocity c has no effect behind frame of this particular triangle through that point whose sides are characteristic curves.

“This behavior is very different from the solution for the heat equation, where the effect of a point source appears (with small amplitude) instantaneously at every point in space.”[1]

In case when t is negative, we have valid solution from above and it is clear that our solution depends on the fact about well posed wave equation Cauchy problem describes, both directions, in forward or backward.

1.5. SPHERICAL WAVES

Next will be described usage of diferential equations is spherical waves. One fact is about waves like this and it must be calculated in type of equation like this one is, dependence of this waves from the radil distance f from a central point source. What that means? Well, for these waves there are represented the three dimensional wave eqation below, and here is also satisfied one-dimensional wave equation. First we have this form of equation:

$$u_{tt} = c^2 \left[u_{rr} + \frac{2}{r} u_r \right].$$

It can be transformed into this

$$(ru)_{tt} = c^2 \left[(ru)_{rr} \right],$$

that is equivalent:

$$u(t,r) = \frac{1}{r} \left[F(r-ct) + G(r+ct) \right],$$

As its final and maybe most understandable version. Here are F and G arbitrary function as it can be assumed. In case when G is zero, than radiation from an antenta fit to this case. Here can be concluded that here are no distortion in time when wave form is transmited from an antenna. In case when are present two spatial dimensions, the feature of undistorted propagation of waves is not showed.[1]

1.6. LAPLACE EQUATION IN TWO DIMENSIONS.

For an unknown function of two variables, given form for φ in Laplace equation is below:

$$\varphi_{xx} + \varphi_{yy} = 0.$$

Well known, harmonic functions, are solutions for these equations. [1]

1.6.1. Connection with holomorphic functions

We cannot separate the Laplace equation and analytic functions of a complex variable because they are tightly related in solutions and therefore describing some complex variable we have the real and imaginary parts of any analytic function that are conjugate harmonic functions.

Satisfying the Laplace equation, we can use gradients in orthogonal form and so, f represents value of u=v. After this state of the Caucht Riemann equation is as below:

$$u_x = v_y, \quad v_x = -u_y,$$
$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0.$$

"Conversely, given any harmonic function in two dimensions, it is the real part of an analytic function, at least locally." If we seek for details, there is Laplace equation. [1]

1.6.2. **Problem of boundary**

It is common for Laplace's equations and solution must be presented in the way to fulfill arbitrary values on the boundary of a domain. What that means? Given is example that obtain harmonic function and take the value $u(\theta)$ on a circle of radius in value 1.

$$\varphi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\theta')}u(\theta')d\theta'.$$

It is Poisson solution and it is fulfilled later with some datas from Petrovsky research, By Petrovsky, formula above can be easily introduced with Fourier series for value of φ.

In case when r is less than 1, next derivates will be easily computed into differenting under the integral sign. Here can be verified value of φ, that is analytic. This is tsable even in case when u represents continuous and not necessary differentiable.

Solutions like elliptic partial differential equations obtain these data.

" The solutions may be much more smooth than the boundary data. This is in contrast to solutions of the wave equation, and more general hyperbolic partial differential equations, which typically have no more derivatives than the data."[1]

1.7. EULER–TRICOMI EQUATION

In the investigation of transonic flow we can use equation invented by the Euler and Triconi, called Euler-Tricomi equation:

$$u_{xx} = xu_{yy}.$$

1.8. ADVECTION EQUATION

The transport of a conserved scalar ψ in a velocity field $\mathbf{u} = (u, v, w)$ is represented by this fallowing equation or we can say, advection equation. It is:

$$\psi_t + (u\psi)_x + (v\psi)_y + (w\psi)_z = 0.$$

If case when the velocity field is solenodial (that is, $\nabla \cdot \mathbf{u} = 0$), then the equation may be simplified to

$$\psi_t + u\psi_x + v\psi_y + w\psi_z = 0.$$

In the one-dimensional case where *u* is not constant and is equal to ψ, the equation is referred to as Burgers' equation.

1.9. GINZBURG–LANDAU EQUATION

For modelling superconditivity we use the Ginzburg-Landau equation. It is:

$$iu_t + pu_{xx} + q|u|^2u = i\gamma u$$

where $p, q \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ are constants and *i* is the imaginary unit.[1]

1.10. THE DYM EQUATION

The Dym equation is named fo Harry Dym and occurs in the study of solitons. It is

$$u_t = u^3 u_{xxx}.$$

1.11. VIBRATING STRING

Stretching the string between two points where $x=0$ and $x=L$ and u denotes the amplitude of the displacement of the string, it can easily serve for the one-dimensional wave equation in the region where $0<x<L$ and t is unlimited. Circumstances obtain case when string is tied down at the ends so boundary conditions can be presented on this way in order to answer on conditions:

$$u(t,0) = 0, \quad u(t,L) = 0,$$

Here as mentioned, initial conditions:

$$u(0,x) = f(x), \quad u_t(0,x) = g(x).$$

As for method, separating variables from wave\s variables can be done through:

$$u_{tt} = c^2 u_{xx},$$

So further, solution will be:

$$u(t,x) = T(t)X(x),$$

Here is , with constant value of k and determined value, equation presented as:

$$T'' + k^2c^2T = 0, \quad X'' + k^2X = 0,$$

The boundary conditions then imply that *X* is a multiple of sin *kx*, and *k* must have the form

$$k = \frac{n\pi}{L},$$

Here is n an integer. So we came to that point. Previous information were needed for better understanding of this fallowing order. where *n* is an integer. Each term in the sum corresponds to a mode of vibration of the string. [1]

The mode with *n*=1 represents the fundamental mode;

The frequencies of the other modes are all multiples of this frequency.

Frequencies will form series that are overtone of the string, and they are the basis for musical acoustics. Next step will be pleasing or fulfilling the initial conditions and best way to do that is by using presentation of f as g and as infinite sums of models like this one is. “Wind instruments typically correspond to vibrations of an air column with one end open and one end closed”. So, the corresponding boundary conditions will be:

$$X(0) = 0, \quad X'(L) = 0.$$

Appling the method of separation in such cases like presented is, is one solution and fallowing it, we will be led to a several odd overtones. [1]

Form solving problems of this type, we can use theory invented by Sturm and Lioville.

1.12. VIBRATING MEMBRANE

Streching a membrane over a curve C that forms the boundary of a domain D in some particular plane, wave equation of the fallowing type can govern these, membranes, vibrations:

$$\frac{1}{c^2}u_{tt} = u_{xx} + u_{yy},$$

if $t>0$ and (x,y) is in D.

The boundary condition is $u(t,x,y) = 0$ if (x,y) is on C. The method of separation of variables leads to the form

$$u(t,x,y) = T(t)v(x,y),$$

which in turn must fulfill next:

$$\frac{1}{c^2}T'' + k^2T = 0,$$
$$v_{xx} + v_{yy} + k^2v = 0.$$

The latter equation is called the Helmholtz Equation. [1]

Allowing a non trivial v in order to fulfill the boundary condition on C, we must determine the constant k. Let's see: k^2 are called the eigenvalues of the Laplacian in D, and the associated solutions are the eigenfunctions of the Laplacian in D. 'The Sturm–Liouville theory may be extended to this elliptic eigenvalue problem' (Jost, 2002).

2. CLASSIFICATION

In order to make some guide for suitable or appropriate boundary conditions, mathematicians made classification among these equations. With usage of such qualification, solutions will come smooth and pretty easy. There are several orders for qualification. For example, for some of linear, second-order partial differential equations classification is as parabolic, hyperbolic or elliptic. Others such as the Euler–Tricomi equation have different types in different regions.

2.1. EQUATIONS OF FIRST ORDER

2.1.1. First-order partial differential equation

Partial differential equation that involves only first derivatives of the unknown function of *n* variables is called a first order partial equation. The equation takes the form

$$F(x_1, \ldots, x_n, u, u_{x1}, \ldots u_{x_n}) = 0.$$

"Such equations arise in the construction of characteristic surfaces for hyperbolic partial differential equations, in the calculus of variations, in some geometrical problems, and they arise in simple models for gas dynamics whose solution involves the method of characteristics."[2]

General solution of integrated families of ordinary equations can be oriented on solution from this single equation of the first order that will later help finding solution for these particular equations. So, we can say that this general solution is obtained into different ordinary differential equations. [2]

2.1.2. Characteristic surfaces for the wave equation

Characteristic surfaces for the wave equation are level surfaces for solutions of the equation

$$u_t^2 = c^2 \left(u_x^2 + u_y^2 + u_z^2\right).$$

In case when wave is set as u_t = 1, there is certain loss of generality and firther will be presented as:

$$u_x^2 + u_y^2 + u_z^2 = \frac{1}{c^2}.$$

In vector notation, let

$$\vec{x} = (x, y, z) \quad \text{and} \quad \vec{p} = (u_x, u_y, u_z).$$

A family of solutions with planes as level surfaces is given by:

$$u(\vec{x}) = \vec{p} \cdot (\vec{x} - \vec{x_0}),$$

where:

$$|\vec{p}| = \frac{1}{c}, \quad \text{and} \quad \vec{x_0} \quad \text{is arbitrary.}$$

'If x and x_0 are held fixed, the envelope of these solutions is obtained by finding a point on the sphere of radius $1/c$ where the value of u is stationary. This is true if $\vec{p}$ is parallel to $\vec{x} - \vec{x_0}$.'[2] Hence the envelope has equation

$$u(\vec{x}) = \pm\frac{1}{c}|\vec{x} - \vec{x_0}\,|.$$

Spheres with arising radius and that have shrinks with velocity c can take this for appropriate solutions. 'These are light cones in space-time' .[2]

As for problems, main issue here will be ordering a level of surface S where u=0 for t=0. If we take the envelope of all the spheres with centers on S, then we will obtain this solution and as for S, it radii will grow with velocity that is noted as c. This envelope is obtained by requiring that

$$\frac{1}{c}|\vec{x} - \vec{x_0}\,| \quad \text{is stationary for} \quad \vec{x_0} \in S.$$

In we put this part, $|\vec{x} - \vec{x_0}\,|$ to relation normal to S, requirements will be pleased. Thus the envelope corresponds to motion with velocity c along each normal to S.

This is the Huygens' construction of wave fronts so each point on *S* will emit a spherical wave at time *t*=0. Than the wave will front at a later time *t* is the envelope of these spherical waves. The normals to *S* are the light rays.[2]

2.2. TWO-DIMENSIONAL THEORY

A general first-order partial differential equation has the form

$$F(x, y, u, p, q) = 0,$$

where

$$p = u_x, \quad q = u_y.$$

A complete integral of this equation is a solution φ(*x*,*y*,*u*) that depends upon two parameters: *a* and *b*.

As for parameter n, it will be required in the n-dimensional case and if we want to develop smooth solution, choosing an arbitrary function w is the way. Later will be settled b, as b=w(a). Next we need to determine A:

- *A*(*x*,*y*,*u*) by requiring that the total derivative

$$\frac{d\varphi}{da} = \varphi_a(x, y, u, A, w(A)) + w'(A)\varphi_b(x, y, u, A, w(A)) = 0.$$

In that case, a solution u_w is also given by

$$u_w = \phi(x, y, u, A, w(A))$$

If we find solution for function w, than it will be easy find solution for our partial differential equation. Another way is 'led to the construction of the light cone as a characteristic surface for the wave equation'. [2]

In a case when a complete integral is not available, solutions may still be obtained by solving a system of ordinary equations. In order to obtain this system, first note that the PDE determines a cone (analogous to the light cone) at each point:

-if the PDE is linear in the derivatives of *u* (it is quasi-linear), then

- the cone degenerates into a line.

In the general case, the pairs (p,q) that satisfy the equation determine a family of planes at a given point:

$$u - u_0 = p(x - x_0) + q(y - y_0),$$

where

$$F(x_0, y_0, u_0, p, q) = 0.$$

The envelope of these planes is a cone, or a line if the PDE is quasi-linear. [2]

So, as for envelope, condition will be:

$$F_p\, dp + F_q\, dq = 0,$$

F -evaluated is at (x_0,y_0,u_0,p,q),

- dp and dq - increments of p and q that satisfy F=0.

Finally, the generator of the cone is a line with fallowing order:

$$dx : dy : du = F_p : F_q : (pF_p + qF_q).$$

Direction c of this generator corresponds to the light rays for the wave equation. In order to integrate differential equations along these directions, we need to find increments for p and q along the ray. It can be further, obtained by differentiating the PDE:

$$F_x + F_u p + F_p p_x + F_q p_y = 0,$$
$$F_y + F_u q + F_p q_x + F_q q_y = 0,$$

Next, the ray direction in (x,y,u,p,q) space will be as below:

$$dx : dy : du : dp : dq = F_p : F_q : (pF_p + qF_q) : (-F_x - F_u p) : (-F_y - F_u q).$$

2.2.1. **Equations of second order**

It we take start, followed by this relation $u_{xy} = u_{yx}$, the general second-order of partial differential equations in two independent variables has the fallowing form:

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + \cdots = 0,$$

Here are the coefficients A, B, C etc. depend upon x and y. If $A^2 + B^2 + C^2 > 0$ over a region of the xy plane, the PDE is second-order in that region. [2]

Same or analogous to this equation form is below relation:

$$Ax^2 + 2Bxy + Cy^2 + \cdots = 0.$$

It is given for a conic section.

By replacing ∂_x by X, for other variables like it is done by a Fourier transform, it will further converts a constant-coefficient of partial differential equation PDE into a polynomial of the same degree, with the top degree or a homogeneous polynomial, here a quadratic form and further, it will be especaily important for our classification.

"Just as one classifies conic sections and quadratic forms into parabolic, hyperbolic, and elliptic based on the discriminant $(2B)^2 - 4AC$, the same can be done for a second-order PDE at a given point. However, the discriminant in a PDE is given by $B^2 - AC$, due to the convention of the xy term being $2B$ rather than B; formally, the discriminant (of the associated quadratic form) is $(2B)^2 - 4AC = 4(B^2 - AC)$, with the factor of 4 dropped for simplicity."[2]

1. $B^2 - AC < 0$: solutions of elliptic PDEs are as smooth as the coefficients allow, within the interior of the region where the equation and solutions are defined. For example, solutions of Laplace's equation are analytic within the domain where they are defined, but solutions may assume boundary values that are not smooth. The motion of a fluid at subsonic speeds can be approximated with elliptic PDEs, and the Euler–Tricomi equation is elliptic where $x<0$.
2. $B^2 - AC = 0$: equations that are parabolic at every point can be transformed into a form analogous to the heat equation by a change of independent variables. Solutions smooth out as the transformed time variable increases. The Euler–Tricomi equation has parabolic type on the line where $x=0$.
3. $B^2 - AC > 0$: hyperbolic equations retain any discontinuities of functions or derivatives in the initial data. An example is the wave equation. The motion of a fluid at supersonic speeds can be approximated with hyperbolic PDEs, and the Euler–Tricomi equation is hyperbolic where $x>0$.

Next, in case we have independent variables for n:

- $x_1, x_2, \ldots, x_n$,

than a general linear partial differential equation of second order will have the fallowing form:

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{plus lower order terms} = 0.$$

'The classification depends upon the signature of the eigenvalues of the coefficient matrix.'

1. Elliptic: The eigenvalues are all positive or all negative.
2. Parabolic : The eigenvalues are all positive or all negative, save one that is zero.
3. Hyperbolic: There is only one negative eigenvalue and all the rest are positive, or there is only one positive eigenvalue and all the rest are negative.
4. Ultrahyperbolic: There is more than one positive eigenvalue and more than one negative eigenvalue, and there are no zero eigenvalues. [2]

2.2.2. **Systems of first-order equations and characteristic surfaces**

Systems of first order will put as in direction where we going to extend our partial differential equation classification and further it will point that is unknown *u* now a vector with *m* components, and the coefficient matrices A_ν are *m* by *m* matrices for the value:

$$\nu = 1, \dots, n.$$

The partial differential equation will further take this form:

$$Lu = \sum_{\nu=1}^{n} A_\nu \frac{\partial u}{\partial x_\nu} + B = 0,$$

where :

-the coefficient matrices A_ν

-the vector *B* may depend upon *x* and *u*.

If a hypersurface *S* is given in the implicit form

$$\varphi(x_1, x_2, \dots, x_n) = 0,$$

where :

-φ has a non-zero gradient,

-then S is a characteristic surface for the operator L at a given point if the characteristic form vanishes:

$$Q\left(\frac{\partial\varphi}{\partial x_1},\dots,\frac{\partial\varphi}{\partial x_n}\right)=\det\left[\sum_{\nu=1}^{n}A_\nu\frac{\partial\varphi}{\partial x_\nu}\right]=0.$$

"The geometric interpretation of this condition is as follows: if data for u are prescribed on the surface S, then it may be possible to determine the normal derivative of u on S from the differential equation. If the data on S and the differential equation determine the normal derivative of u on S, then S is non-characteristic. If the data on S and the differential equation *do not* determine the normal derivative of u on S, then the surface is characteristic, and the differential equation restricts the data on S: the differential equation is *internal* to S."

1. A first-order system $Lu=0$ is *elliptic* if no surface is characteristic for L: the values of u on S and the differential equation always determine the normal derivative of u on S.
2. A first-order system is *hyperbolic* at a point if there is a space-like surface S with normal ξ at that point. This means that, given any non-trivial vector η orthogonal to ξ, and a scalar multiplier λ, the equation

$$Q(\lambda\xi+\eta)=0,$$

has m real roots $\lambda_1, \lambda_2, ..., \lambda_m$.

Roots are distinct and as for the system, it is hyperbolic.

The geometrical interpretation of this condition is as follows: "the characteristic form$Q(\zeta)=0$ defines a cone (the normal cone) with homogeneous coordinates ζ. In the hyperbolic case, this cone has m sheets, and the axis $\zeta = \lambda\,\xi$ runs inside these sheets: it does not intersect any of them."

Displaced from the origin by η, axis will intersect every sheet. In the elliptic case, the normal cone has no real sheets.

2.2.3. Equations of mixed type

Partial differential equations have unstable coefficients and that is why can not be part of any other category rather then mixed type. In other words, this particular coefficient of our partial differential equation is going to be included in mixed type of equation.

Here is given one simple example: the Euler–Tricomi equation called elliptic-hyperbolic because it is elliptic in the region *x* < 0, hyperbolic in the region *x* > 0. Below:

$$u_{xx} = xu_{yy}$$

It degenerates parabolic on the line *x* = 0.

2.2.4. Infinite-order PDEs in quantum mechanics

Quantum Hamilton's equations for trajectories of quantum particles is led by a Weyl quantization in phase space . Equations like theseare infinite-order partial differential equation. In the semiclassical expansion one has a finite system of ODEs at any fixed order of $\hbar$. 'The equation of evolution of the Wigner function is infinite-order PDE also. The quantum trajectories are quantum characteristics with the use of which one can calculate the evolution of the Wigner function.' [2]

3. SOLVING ANDE ANALYZING PDES

3.1. INTEGRAL TRANSFORM

Partial differential equation can be transformed into simplier form or separable form of partial differential equation. It will be according to diagnolizing an operator as it is.

Example for this way is Fourier analysis. It diagonalizes the heat equation using the eigenbasis of sinusoidal waves.

“If the domain is finite or periodic, an infinite sum of solutions such as a Fourier series is appropriate, but an integral of solutions such as a Fourier integral is generally required for infinite domains. The solution for a point source for the heat equation given above is an example for use of a Fourier integral.” [2]

3.2. CHANGE OF VARIABLES

Reducing partial differential equation in one more simplier form is done through solution of adaptable variable’s changes There is a nice example of changing these varibels in equation invented by Black and Scholes, as below:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

It is reducible to the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

$$\begin{aligned} V(S,t) &= Kv(x,\tau) \\ x &= \ln(S/K) \\ \tau &= \frac{1}{2}\sigma^2(T-t) \\ v(x,\tau) &= \exp(-\alpha x - \beta\tau)u(x,\tau). \end{aligned}$$

3.3. LIE GROUP METHODS

Another theory of differential equation was invented by a Sophus L in year of 1870. It was more pleasing form of equation for finding solution. “He showed that the integration theories of the older mathematicians can, by the introduction of what are now called Lie groups, be referred to a common source; and that ordinary differential equations which admit the same infinitesimal transformations present comparable difficulties of integration." Another thing he did, is emphasizing the subject of transformatios of contant.[2]

Symmetry property of differential equations is used for another solution ways, with the continuous infinitesimal transformations of solutions to solutions by the Lie theory.

By a cContinuous group theory, in order to understand the structure of linear and nonlinear partial differential equations, used are Lie algebras and differential geometry and also in rder to for generate inferable equations, so as for finding its

- -Lax pairs, recursion operators,
- Bäcklund transform and finally
- finding exact analytic solutions to the partial differential equations.

In studing study differential equations many symmetry methods are used , in a different field of activities, for example in mathematics, physics, engineering, and so as in other disciplines.

3.4. NUMERICAL METHODS TO SOLVE PDES

In order to solve partial differential equations , most common in usage of the three numerical methods:

1. the finite element method (FEM),
2. finite volume methods (FVM) and
3. finite difference methods (FDM).

"The FEM has a prominent position among these methods and especially its exceptionally efficient higher-order version hp-FEM. "Other versions of FEM include

- the generalized finite element method (GFEM),

-extended finite element method (XFEM),

-spectral finite element method (SFEM),

- meshfree finite element method,

-discontinuous Galerkin finite element method (DGFEM), etc.

3.4.1. **Finite Element Method**

Numerical technique for finding approximate solutions of a partial differential equations and integral equations in the finite elemente method, known as FEA. It is a practical application often known as finite element analysis or as it is said, FEA. Finding solution is in direction of eliminating the differential equation steady state problem or by rendering these partial differential equations into approximating system of ordinary differential equation. These equations will be further numerically integrated by using standard techniques. For example there are some techniques like Euler's method, Runge-Kutta, etc.[2]

3.4.2. **Finite Difference Method**

For approximating the solutions to differential equations using finite difference equations derivatives we use so called finite difference methods.

3.4.3. **Finite Volume Method**

"Finite volume" refers to the small volume surrounding each node point on a mesh. Similar to the finite difference method or finite element method, we can calculate values in discrete places on a meshed geometry. In the finite volume method, “volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem; these terms are then evaluated as fluxes at the surfaces of each finite volume”. These methods are conservative because of the flux that goes into a volume given and identical to the leaving adjacent volume.

4. EXAMPLES OF IMPORTANT PARTIAL DIFFERENTIAL EQUATIONS THAT ARISE IN PROBLEMS OF MATHEMATICAL PHYSICS

Benjamin-Bona-Mahony equation

$$u_1 + u_x + u\, u_x - u_{x\,x\,1} = 0$$

Biharmonic equation

$$\nabla^4 \varphi = 0$$

Boussinesq equation

$$u_{tt} + \alpha\, u_{xx} + \beta (u^2)_{xx} + \gamma\, u_{xxxx} = 0$$

Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

Chaplygin's equation

$$u_{xx} + \frac{y^2}{1 - \frac{y^2}{c^2}}\, u_{yy} + y\, u_y = 0.$$

Euler-Darboux equation

$$u_{xy} + \frac{\alpha\, u_x - \beta\, u_y}{x - y} = 0.$$

Heat conduction equation

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) &= 0, \\
\frac{\partial (\rho \mathbf{V})}{\partial t} + \nabla \cdot (\rho \mathbf{V}\mathbf{V} + p\mathbf{I}) &= 0, \qquad (1) \\
\frac{\partial E}{\partial t} + \nabla \cdot (E + p)\mathbf{V} &= \nabla \cdot (\kappa \nabla T),
\end{aligned}$$

Helmholtz differential equation

$$(\nabla^2 + k^2)\phi = 0$$

Klein-Gordon equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0,$$

Korteweg-de Vries-Burgers equation

$$u_t + 2\, u\, u_x - \nu\, u_{xx} + \mu\, u_{xxx} = 0.$$

Korteweg-de Vries equation

$$u_t + u_{xxx} - 6\, u\, u_x = 0.$$

Krichever-Novikov equation

$$\frac{u_t}{u_x} = \frac{1}{4}\frac{u_{xxx}}{u_x} - \frac{3}{8}\frac{u_{xx}^2}{u_x^2} + \frac{3}{2}\frac{p(u)}{u_x^2},$$

where

$$p(u) = \tfrac{1}{4}\left(4u^3 - g_2 u - g_3\right).$$

Laplace's equation

$$\nabla^2 \psi = 0.$$

Lin-Tsien equation

$$2u_{tx} + u_x u_{xx} - u_{yy} = 0.$$

Sine-Gordon equation

$$v_{tt} - v_{xx} + \sin v = 0.$$

Spherical harmonic differential equation

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + l(l+1)\right]u = 0.$$

Tricomi equation

$$u_{yy} = y\, u_{xx}.$$

Wave equation

5. SINGLE PARTIAL DIFERENTIAL EQUATIONS

a. Linear equations.

1. *Laplace's equation*

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0.$$

2. *Helmholtz's (or eigenvalue) equation*

$$-\Delta u = \lambda u.$$

3. *Linear transport equation*

$$u_t + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

4. *Liouville's equation*

$$u_t - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

5. *Heat (or diffusion) equation*

$$u_t - \Delta u = 0.$$

6. *Schrödinger's equation*

$$iu_t + \Delta u = 0.$$

7. *Kolmogorov's equation*

$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

8. *Fokker–Planck equation*

$$u_t - \sum_{i,j=1}^{n} (a^{ij} u)_{x_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i} = 0.$$

9. *Wave equation*

$$u_{tt} - \Delta u = 0.$$

10. *Klein–Gordon equation*

$$u_{tt} - \Delta u + m^2 u = 0.$$

11. *Telegraph equation*

$$u_{tt} + 2du_t - u_{xx} = 0.$$

$$u_{tt} + 2du_t - u_{xx} = 0.$$

12. *General wave equation*

$$u_{tt} - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0.$$

13. *Airy's equation*

$$u_t + u_{xxx} = 0.$$

14. *Beam equation*

$$u_{tt} + u_{xxxx} = 0.$$

b. Nonlinear equations.

1. *Eikonal equation*

$$|Du| = 1.$$

2. *Nonlinear Poisson equation*

$$-\Delta u = f(u).$$

3. *p-Laplacian equation*

$$\mathrm{div}(|Du|^{p-2} Du) = 0.$$

4. *Minimal surface equation*

$$\operatorname{div}\left(\frac{Du}{(1+|Du|^2)^{1/2}}\right)=0.$$

5. *Monge–Ampère equation*

$$\det(D^2u)=f.$$

6. *Hamilton–Jacobi equation*

$$u_t+H(Du,x)=0.$$

7. *Scalar conservation law*

$$u_t+\operatorname{div}\mathbf{F}(u)=0.$$

8. *Inviscid Burgers' equation*

$$u_t+uu_x=0.$$

9. *Scalar reaction-diffusion equation*

$$u_t-\Delta u=f(u).$$

10. *Porous medium equation*

$$u_t-\Delta(u^\gamma)=0.$$

11. *Nonlinear wave equation*

$$u_{tt}-\Delta u+f(u)=0.$$

12. *Korteweg–deVries (KdV) equation*

$$u_t+uu_x+u_{xxx}=0.$$

13. *Nonlinear Schrödinger equation*

$$iu_t+\Delta u=f(|u|^2)u.$$

[3]

6. SYSTEMS OF PARTIAL LINEAR EQUATIONS

Linear systems.

1. *Equilibrium equations of linear elasticity*

$$\mu\Delta\mathbf{u} + (\lambda + \mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

2. *Evolution equations of linear elasticity*

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda + \mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0}.$$

3. *Maxwell's equations*

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B} \\ \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0. \end{cases}$$

Nonlinear systems.

1. *System of conservation laws*

$$\mathbf{u}_t + \operatorname{div}\mathbf{F}(\mathbf{u}) = \mathbf{0}.$$

2. *Reaction-diffusion system*

$$\mathbf{u}_t - \Delta\mathbf{u} = \mathbf{f}(\mathbf{u}).$$

3. *Euler's equations for incompressible, inviscid flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

4. *Navier–Stokes equations for incompressible, viscous flow*

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot D\mathbf{u} - \Delta\mathbf{u} = -Dp \\ \operatorname{div}\mathbf{u} = 0. \end{cases}$$

[3]

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BIOGRAPHY

Abear Saeed Aboglida is born 17.2.1986. in Ajilat in Libya. High school "Agelat" finishes in 2003. and enrolles at University of Agelat which finishes the 2007th year.

At Faculty of Sciences at University of Novi Sad, Department of Mathematics and Informatics, master of mathematics, enroles in 2009. as a grant student of Libian Goverment.

At Master studies she has passed all exams that are specified in curriculum. She speaks english.

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AB
PDE is shorter name for partial differential equations ,which have so many usages in different fields of acting. It is type of differential equation.

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