UNIVERSITY OF NOVI SAD
FACULTY OF SCIENCES
DEPARTMENT OF
MATHEMATICS AND INFORMATICS

Marija Jovanović

L-COLORINGS OF GRAPHS

- master thesis -

Mentor:

dr Vlado Uljarević

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Abstract

Graph coloring is one of the most widely appreciated and valuable aspects within the field of graph theory. Graph coloring involves the allocation of integers to the various vertices of a graph in such a way that no two adjacent vertices receive the same integer value. This particular challenge frequently emerges in scenarios involving scheduling and channel allocation. In the context of a graph, a list coloring is an arrangement of integers assigned to its vertices, but with an added constraint that these integers must originate from predetermined lists of viable colors associated with each vertex.

In this thesis, we give an overview of some interesting results in the list colorings in the past three decades. We introduce the concept of list coloring by giving some examples of applications, then give brief overview of basic terms and results in graph theory that will be used further on. We continue with list colorings, characterization of 2-choosable graphs and planar graphs, explaining the famous Five-Color theorem followed by the Mirzakhani graph. We finish by mentioning list-edge-colorings and proving Galvin's lemma.

Contents

Chapter 1

Introduction

In graph theory, list coloring is a variant of the traditional graph coloring problem, where each vertex in a graph is assigned a color from a specific list of available colors. Unlike standard graph coloring, where a universal set of colors is available to all vertices, list coloring allows for flexibility by giving each vertex its own list of permissible colors. The goal, however, remains the same: to ensure that adjacent vertices have distinct colors.

The concept of list coloring was introduced by Vizing [22] in the 1970s and independently by Erdős, Rubin, and Taylor [8], and it generalizes traditional graph coloring. It provides a deeper insight into the behavior of graphs, particularly when constraints are placed on available colors at each vertex. This extension is useful in real-world applications where certain limitations or restrictions exist, such as scheduling problems, frequency assignments in wireless networks, or timetabling, where different entities might have limited resources or preferences.

In traditional graph coloring, the same set of colors is available to all vertices. However, list coloring accounts for real-life constraints where resources (colors) available to one entity may not be available to another. This makes list coloring a more nuanced and flexible model, applicable to diverse scheduling and resource allocation problems.

In addition to its practical applications, list coloring has deep theoretical importance in the study of graph algorithms and complexity, often intersecting with problems in combinatorics, probability, and optimization.

The study of list coloring also intersects with various classical results in graph theory, such as those concerning planar graphs, bipartite graphs, and other special graph classes. For instance, certain theorems that hold in classical graph coloring, like the Four Color Theorem for planar graphs, have analogous or modified counterparts in the context of list coloring, though with often higher complexity and technical challenges. The behavior of graphs under list coloring constraints provides insight into their combinatorial structure, shedding light on how localized restrictions can impact global properties.

Another significant aspect of list coloring is its inherent complexity. Unlike classical graph coloring, which is already computationally hard, list coloring adds another layer of difficulty due to the individualized nature of the constraints. The problem remains NP-complete in general, making it a challenging area for algorithmic graph theory. Researchers have explored various heuristic and approximate methods for finding list colorings in practice, particularly in cases where exact solutions are computationally infeasible. Despite the difficulty of the problem, list coloring provides a powerful model for solving practical problems in areas such as scheduling, network design, and resource allocation, where constraints vary across the system.

In conclusion, list coloring represents a significant extension of graph coloring theory, introducing additional complexity and flexibility that better captures many real-world scenarios. Its connections to other areas of mathematics, its challenging computational aspects, and its relevance to practical problems all contribute to its importance as a field of study in modern graph theory. As research in this area continues, new results and techniques are likely to further deepen our understanding of the interplay between local constraints and global graph properties.

1.1 Applications

List coloring is a variation of graph coloring, where each vertex of a graph is assigned a list of colors instead of a single color. List coloring has numerous applications in graph theory and computer science. It is used to solve problems such as the scheduling of courses, assigning frequencies to radio stations to avoid interference, and allocating resources in computer networks. We mention some of its applications listed below.

Wireless Channel Assignment [17]: List coloring is used in wireless communication systems to assign channels to devices to avoid interference. Each device is assigned a list of available channels, and the goal is to assign channels to devices in a way that neighboring devices have distinct channels. List coloring algorithms can be employed to find efficient channel assignments, ensuring minimal interference and maximizing network performance.

Register Allocation [15]: List coloring plays a crucial role in compiler optimization, specifically in the register allocation phase. In compilers, registers are limited resources used to store intermediate values during program execution. List coloring is employed to allocate registers to variables, ensuring that no two variables sharing the same scope are assigned the same register. By using list coloring techniques, compilers can optimize register usage and improve the overall performance of compiled programs.

Resource Allocation in Timetable Scheduling[2]: List coloring can be applied to timetable scheduling problems, where a set of resources, such as classrooms, is allocated to a set of activities, such as classes or exams. Each resource is assigned a list of available time slots, and list coloring algorithms can be used to assign time slots to activities, ensuring that no conflicting activities are scheduled simultaneously in the same resource.

Frequency Assignment in Wireless Networks [19]: List coloring is used in frequency assignment problems in wireless networks. In cellular networks, different cells are allocated specific frequencies to enable communication. However, adjacent cells need to use different frequencies to avoid interference. List coloring algorithms are utilized to allocate frequencies to cells, ensuring that neighboring cells have distinct frequencies and minimizing signal interference.

Coloring Maps: List coloring techniques have applications in map coloring problems. When coloring a map, the goal is to assign colors to regions such that neighboring regions have distinct colors. List coloring algorithms can be employed to determine the minimum number of colors required to color a given map, which is known as the chromatic number of the map. This problem has practical applications in areas such as cartography and geographic information systems.

TASK SCHEDULING: List coloring can be used in task scheduling problems, where a set of tasks needs to be scheduled on a set of resources with certain constraints. Each task is assigned a list of available time slots or resources, and list coloring algorithms can be employed to find feasible schedules that satisfy the given constraints. This has applications in project management, job scheduling, and resource allocation in various domains.

BIOINFORMATICS AND BIOLOGICAL NETWORKS [14]: In bioinformatics, list colorings are used to analyze biological networks like protein interaction networks and genetic interaction networks. List colorings help identify patterns and relationships within complex biological data, aiding in disease research and biomarker identification.

SUDOKU [13]: List coloring can be applied to solve Sudoku puzzles. In a Sudoku puzzle, a 9x9 grid is divided into nine 3x3 subgrids, and the goal is to fill in the grid such that each row, column, and subgrid contains the numbers 1 to 9 without repetition. List coloring can be used to assign a list of possible numbers to each cell based on the existing numbers in the row, column, and subgrid. The puzzle can then be solved by iteratively reducing the lists until each cell contains a single number.

These are just a few examples of the applications of list coloring. The technique finds use in a wide range of fields, including mathematics, computer science, telecommunications, and logistics, among others.

1.2 Mathematical preliminaries

1.2.1 Definitions of graphs

A **graph** G is an ordered pair $(V(G), E(G))$, where nonempty set $V(E)$ is a **set of vertices** and a possibly empty set $E(G)$ of 2-element subsets of $V(E)$ is a **set of edges**. The vertex set of G is denoted $V(G)$, the edge set $E(G)$, while the number of vertices and the number of edges we denote by $|V(G)|$ and $|E(G)|$, respectively. If $\{u, v\}$, or shorter just uv or vu, is an edge of G, then vertices u and v are **adjacent vertices**. Two adjacent vertices are called **neighbors** of each other. The set of neighbors of a vertex v is called the **open neighborhood** of v or just the **neighborhood** of v and is denoted by $N_G(v)$ or just $N(v)$ if the graph G is understood. The set $N[v] = N(v) \cup v$ is called the **closed neighborhood** of v. If uv and vw are distinct edges in G, then uv and vw are **adjacent edges**. The vertex u and the edge uv are said to be **incident** with each other.

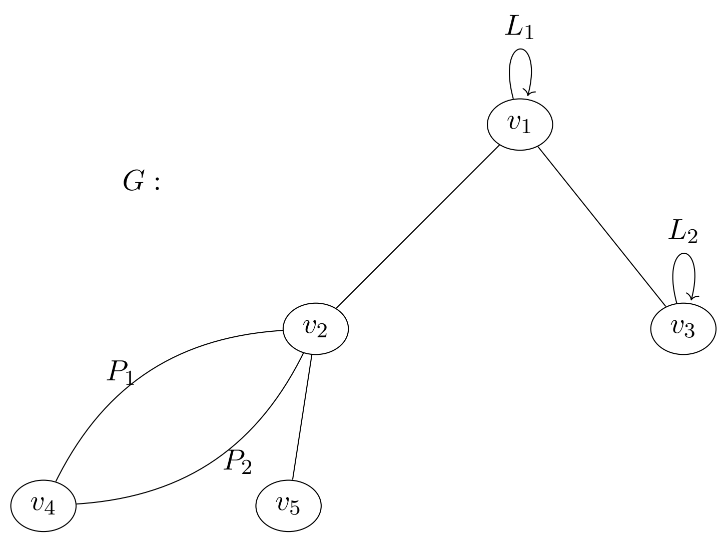


Figure 1.2.1: The graph G with loops and parallel edges

The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. A graph of order 1 is called a **trivial graph**. A **nontrivial graph** therefore has two or more vertices. A graph of size 0 is called an **empty graph** or a **null set**. A nonempty graph then has one or more edges. In any empty graph, no two vertices are adjacent. An edge with identical ends is called a **loop** and an edge with distinct ends a **link**. Two or more links with the same pair of ends are called **parallel edges**. A graph is **simple** if it has no loops or parallel edges. The **degree of a vertex** of a graph is the number of edges that are incident to the vertex, denoted by $d(v)$. The **maximum degree** of a graph G is denoted by $\Delta(G)$, and is the maximum of G's vertices' degrees. The **minimum degree** of a graph is denoted by $\delta(G)$ and is the minimum of G's vertices' degrees. A **regular graph** is a graph where each vertex has the same number of neighbors, that is, every vertex has the same degree, so we can speak of the degree of the graph.

Example 1.2.1. In Figure 1.2.1, graph G has loops L_1 and L_2, and parallel edges P_1 and P_2. △

Example 1.2.2. Figure 1.2.2 shows a graph G with maximum degree $\Delta(G) = 3$, at vertex v_2, and minimum degree $\delta(G) = 1$, at vertex v_6. △

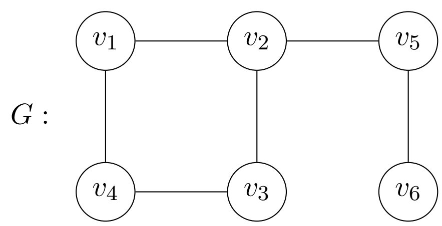


Figure 1.2.2: A graph G

Example 1.2.3. In Figure 1.2.3 graph G is a regular graph which has a degree 2.

△

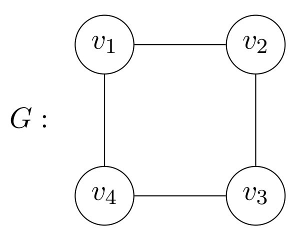


Figure 1.2.3: The regular graph G

A **complete graph** is a simple graph in which any two vertices are connected by a unique edge. We denote it by K_n, where n is the number of vertices in the graph. K_n is a special kind of regular graph where all vertices have the maximum possible degree, $n-1$.

Example 1.2.4. Graph K_4, shown in Figure 1.2.4, is both regular and complete with $n = 4$ and $\Delta(G) = 3$. △

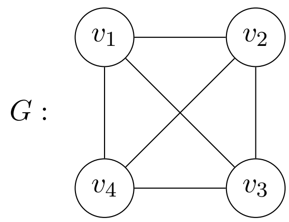


Figure 1.2.4: The complete graph K_4

A **path** is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence and are nonadjacent otherwise.

Example 1.2.5. An example of a path is a graph G_3 showed in Figure 1.2.7.

△

A **cycle graph** or **circular graph** is a graph that consists of a single cycle, or in other words, some number of vertices (at least 3, if the graph is simple) connected in a closed chain. The cycle graph with n vertices is denoted $C_n = v_1v_2 \ldots v_nv_1$. The number of vertices in C_n equals the number of edges, and every vertex has degree 2, that is, every vertex has exactly two edges incident with it. A cycle on one vertex consists of a single vertex with a loop and a cycle on two vertices consists of two vertices joined by a pair of parallel edges. The length of a path or a cycle is the number of its edges. An **acyclic graph** is one that contains no cycles.

Example 1.2.6. A cycle C_4 is showed in Figure 1.2.3. △

A **planar graph** is a graph that can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. Such a drawing is also called an **embedding** of G in the plane. In this case, the embedding is a planar embedding. A graph G that is already drawn in the plane in this manner is a **plane graph**. A

face of the graph is a region bounded by a set of edges and vertices in the embedding. Note that in any embedding of a graph in the plane, the faces are the same in terms of the graph, though they may be different regions in the plane. The face with unbounded area is known as the **unbounded face**, the **outer face**, or the **infinite face** and other faces are its **inner faces**. A **near-triangulation** is a plane graph all of whose inner faces are of degree three.

Example 1.2.7. A graph in the Figure 1.2.3 is planar, while the one in the Figure 1.2.10 is not. △

An **interval graph** is an undirected graph formed from a set of intervals on the real line, with a vertex for each interval and an edge between vertices whose intervals intersect. It is the intersection graph of the intervals. Each vertex of the interval graph can be associated with an interval on the real line in such a way that two vertices are adjacent if and only if the associated intervals have a nonempty intersection. These intervals are said to form an interval representation of the graph. We denote by I the property of being an interval graph.

1.2.2 Graph isomorphism

A **graph homomorphism** f from a graph $G = (V(G), E(G))$ to a graph $H = (V(H), E(H))$, $f : V(G) \to V(H)$, is a function from $V(G)$ to $V(H)$ such that if $(u, v) \in E(G)$, then $(f(u), f(v)) \in E(H)$, for all pairs of vertices $u, v \in V(G)$. An **isomorphism of graphs** G and H is a bijection between the vertex sets of G and H $f : V(G) \to V(H)$ such that any two vertices u and v of G are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in H.

Example 1.2.8. Graphs G and H shown in Figure 1.2.5 are isomorphic with isomorphism f, defined as $f(v_i) = u_i, i = 1, 2, \ldots, 6$, despite their different looking drawings.

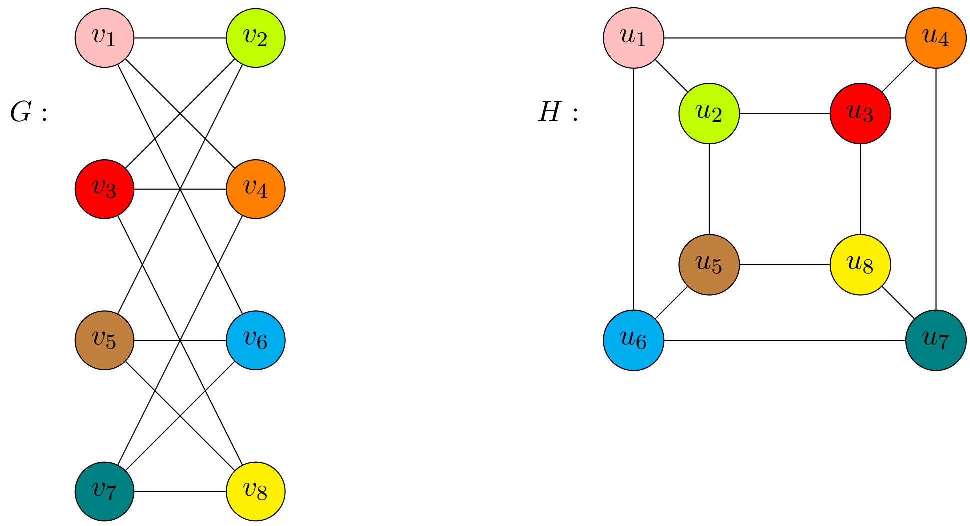


Figure 1.2.5: An isomorphism $f : V(G) \to V(H)$

$\triangle$

If graphs G and H are isomorphic, we denote it by $G \cong H$.

Example 1.2.9. Let G be a graph, like one in Figure 1.2.6(a) and G' a graph like the one in Figure 1.2.6 (b). There exists mapping $f : G \to G'$ such that if $(u, v) \in E(G)$, then $(f(u), f(v)) \in E(G')$, for all $u, v \in V(G)$. Let us define that $f(a) = x$, $f(b) = y$, $f(c) = z$, $f(d) = x$, $f(e) = z$.

If (a, b) is an edge in G, then $(f(a), f(b))$ must be an edge in G' since $f(a) = x$, $f(b) = y$ implies $(f(a), f(b)) = (x, y) \in E(G')$.

If (b, c) is an edge in G, then $(f(b), f(c))$ must be an edge in G' since $f(b) = y$, $f(c) = z$ results in $(f(b), f(c)) = (y, z) \in E(G')$.

If (c, d) is an edge in G, then $(f(c), f(d))$ must be an edge in G' since $f(c) = x$, $f(d) = x$ leads to $(f(c), f(d)) = (z, x) \in E(G')$.

For edges (d, e),$(e, a) \in G$ similarly we get that $(f(d), f(e)), (f(e), f(a)) \in G'$, so, since all edges from G are preserved in graph G', f is a homomorphism. $\triangle$

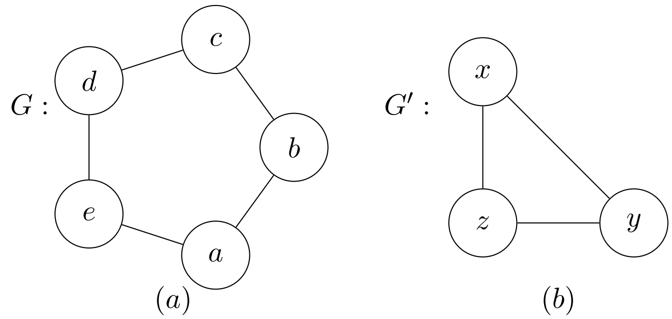


Figure 1.2.6: A homomorphism $f : V(G) \to V(H)$

1.2.3 Subgraphs

A graph H is a **subgraph** of a graph $G = (V, E)$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, we denote it as $H \subseteq G$. If $H \subseteq G$, then G is a **supergraph** of H. If $V(H) = V(G)$ then H is a **spanning subgraph** of G. In other words, a spanning subgraph is obtained by edge deletions only. If H is a subgraphs of G and $G \neq H$, then H is a **proper subgraph** of G. There are two natural ways of deriving subgraphs from G. If $e \in E$, $|E(G)| = m$, we may obtain a graph on $m-1$ edges by deleting e from G, but leaving the vertices and the remaining edges intact. We denote the resulting graph by $G\backslash e$ and call it an **edge-deleted subgraph**. Similarly, if v is a vertex of G, $|V(G)| = n$, we may obtain a graph on $n-1$ vertices by deleting the vertex v from G, together with all the edges incident with v. The resulting graph is denoted by $G - v$ and called a **vertex-deleted subgraph**. More generally, if $V' = V(G')$, $V' \subseteq V$, the **difference between two graphs** G and G', $G - G'$ is the remaining subgraph H of G after the subgraph G' is removed from G; subgraph H is the graph G with vertex set $V \backslash V'$.

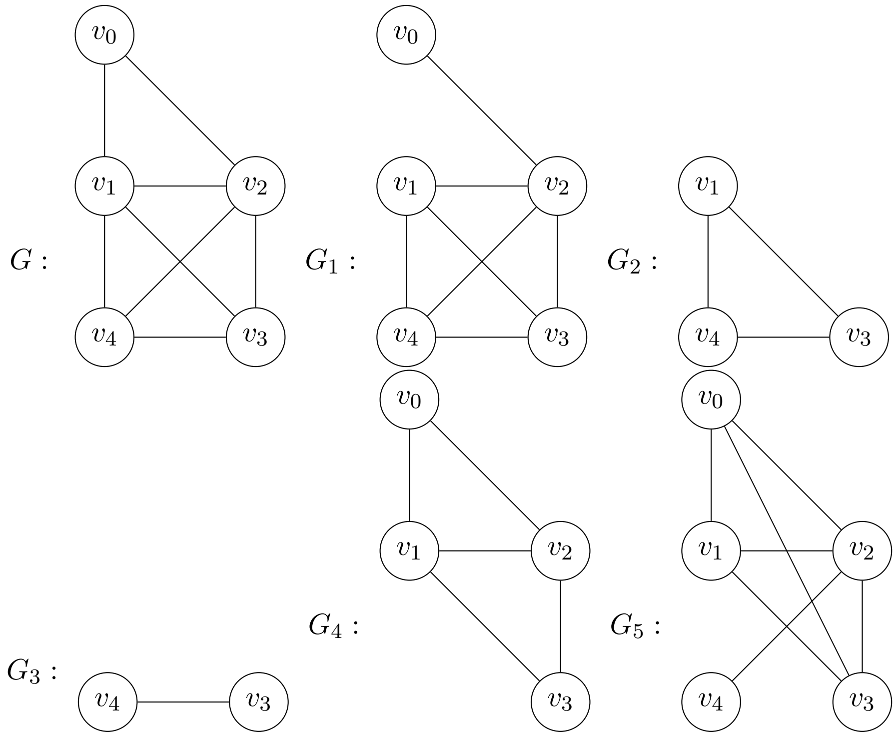


Figure 1.2.7: Graphs and subgraphs

Example 1.2.10. In the Figure 1.2.7 we have graph G and its proper subgraphs G_1, G_2, G_3, G_4. Graph G is a subset of itself, but it is not a proper subgraph of G. Graph G_5 is not a subgraph of G since it contains an edge v_0v_3 and this is not an edge in G. The graph G_1 is a spanning subgraph of G because $V(G) = V(G_1)$. An example of the edge-deleted subgraphs is $G_1 = G\backslash v_0v_1$ and an example of a vertex-deleted subgraph is $G_4 = G - v_4$. $\triangle$

A subgraph H of G is called a **core** of G if there is a homomorphism $f : G \to H$, but no homomorphism $f : G \to H'$ for any proper subgraph H' of H. A graph which is its own core will be called simply a core. Any complete graph is a core. A cycle of odd length is a core.

1.2.4 Connectivity

A non-empty graph $G = (V, E)$ is **connected** if any two of its vertices are linked by a path in G. If $U \subseteq V(G)$ and $G(U)$ is connected, then U is itself connected in G. A maximal connected subgraph of G is called a

component of G. Component is always a non-empty graph, the empty graph has no components.

Example 1.2.11. A graph with three connected components is shown in Figure 1.2.8.

△

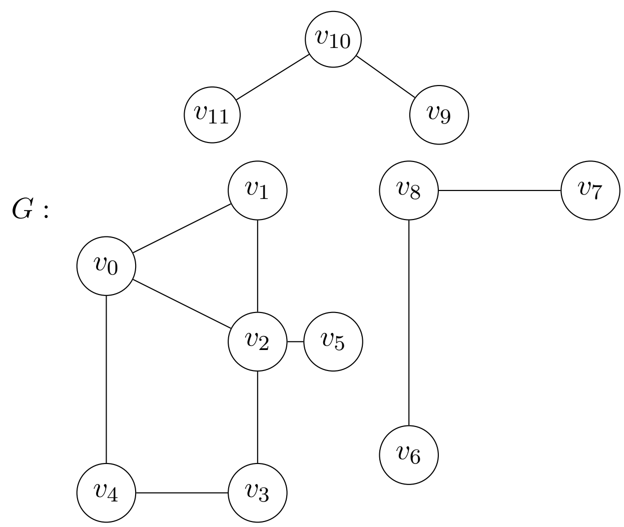


Figure 1.2.8

If there is more than one connected component for a given graph, then the union of connected components will give the set of all vertices of the given graph. Connected component sets are pairwise disjoint. Recall that it means if we take the intersection between two different connected component sets then the intersection will be equal to an empty set or a null set. If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every $A - B$ path in G contains a vertex of an edge form X, then it is said that X **separates** the sets A and B in G. From this it follows that $A \cap B \subseteq X$. If X separates two vertices for $G - X$ in G, then X is called a **separating set** in G. A vertex which separates two other vertices of the same component is called a **cutvertex** and an edge separating its ends is a **bridge**. Note that the bridges in a graph do not lie on any cycle. Graph G is said to be $k-$**connected**, where $k \in \mathbb{N}$, if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. In other words, no two vertices are separated by fewer than k other vertices. Every non-empty graph is 0-connected and the 1-connected graphs are the non-trivial connected graphs. The greatest integer k such that graph G is

k-connected is called the **connectivity** $k(G)$ of G. Note that $k(G) = 0$ if and only if G is disconnected or K^1, and $k(K^n) = n - 1$, for all $n \geq 1$. A connected acyclic graph is called a **tree**.

Example 1.2.12. A graph with cutvertices v, x and y and bridge xy is shown on Figure 1.2.9.

△

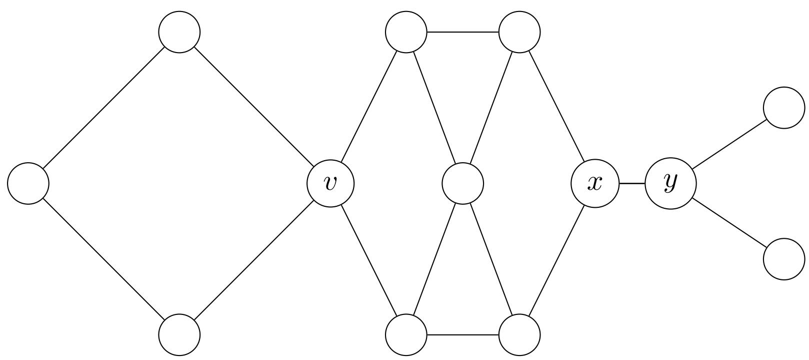


Figure 1.2.9: A graph with cutvertices and a bridge

1.2.5 Bipartite graphs

A graph G is **bipartite** is its vertex set can be partitioned into two subsets, X and Y, so that every end has one end in X and another in Y; such a partition (X, Y) is called a bipartition of the graph and X and Y are called its parts.

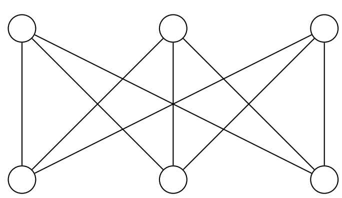


Figure 1.2.10: A bipartite graph

We denote a bipartite graph G with bipartition (X, Y) by $G[X, Y]$. If $G[X, Y]$ is simple and every vertex in X is joined to every vertex in Y, then G is called a **complete bipartite graph**. We denote a complete bipartite graph by $K_{n,m}$, where $n = |X|$ and $m = |Y|$.

Example 1.2.13. A complete bipartite graph $K_{3,3}$ is shown in Figure 1.2.10.

A **star** is a complete bipartite graph $G[X, Y]$ with $|X| = 1$ or $|Y| = 1$. We denote it by $K_{1,n}$, when n is the number of edges of a graph.

Example 1.2.14. A star graph $K_{1,7}$ is shown in Figure 1.2.11. △

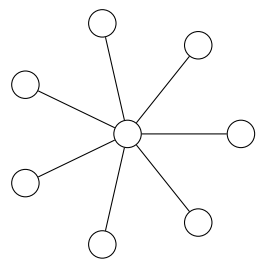


Figure 1.2.11: A star graph

1.2.6 Graph operations

Graph operations are operations which produce new graphs from initial ones. They include both unary (one input) and binary (two input) operations.

One of the most common binary operations defined on graphs is the union of graphs. The **union** $G = G_1 + G_2$ of G_1 and G_2 has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$. The union $G + G$ of two disjoint copies of G is denoted by $2G$. Indeed, if a graph G consists of k (≥ 2) disjoint copies of a graph H, then we write $G = kH$.

$K_4 + G_3$:

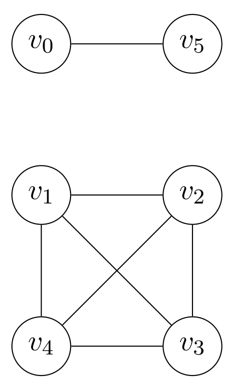


Figure 1.2.12: The union of graphs

Example 1.2.15. The union of graphs K_4 and a path G_3 from Figure 1.2.7 is shown on Figure 1.2.12.

$\triangle$

The most familiar graph operation of a graph is the line graph. The **line graph** $L(G)$ of a graph G is the graph whose vertices can be put in one-to-one correspondence with the edges of G in such a way that two vertices of $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G.

Example 1.2.16. Figure 1.2.13 shows graph G and its line graph $L(G)$.

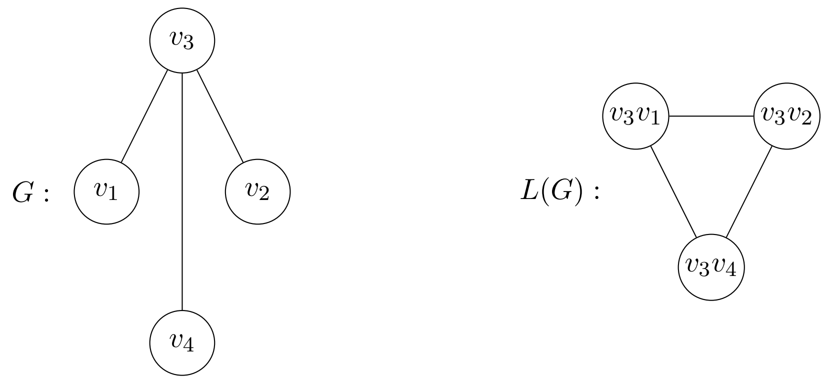


Figure 1.2.13: A graph and its line graph

1.2.7 Power set and finite projective plane

For a set X, the **power set** of X, denoted by $\mathcal{P}(X)$, is the set of all subsets of X.

Obviously, for any set X, we have that $\emptyset \in \mathcal{P}(X)$ and $X \in \mathcal{P}(X)$. Furthermore, if X is finite, then $|\mathcal{P}(X)| = 2^{|X|}$. A **finite projective plane** is set system $(X, \mathcal{P})$ such that X is a finite set, $\mathcal{P}$ power set of X and the following three properties are satisfied:

- (P0) there exists a 4-element subset $Q \subseteq X$ such that every $P \in \mathcal{P}$ satisfies $|P \cap Q| \leq 2$;
- (P1) all distinct $P_1, P_2 \in P$ satisfy $|P_1 \cap P_2| = 1$;
- (P2) for all distinct $x_1, x_2 \in X$, there exists a unique $P \in \mathcal{P}$ such that $x_1, x_2 \in P$.

If $(X, \mathcal{P})$ is a finite projective plane, then members of X are called **points**, and members of $\mathcal{P}$ are called **lines**. For a point $x \in X$ and a line $P \in \mathcal{P}$ such that $x \in P$, we say that the line P is **incident** with the point x, or that P **contains** x, or that P **passes through** x.

1.2.8 Directed graphs

A **directed graph** or a **digraph** D is an ordered pair $(V(D), A(D))$, where $V = V(D)$ is a set of vertices and $A = A(D)$ a set of arcs, together with an **incidence function** $\psi(D)$ that associates with each arc of D an ordered pair of (not necessarily distinct) vertices of D.

If a is an arc and $\psi_{D(a)} = (u, v)$, then a is said to **join** u to v. It is said that u **dominates** v. The vertex u is the **tail** of a, and the vertex v its **head**, they are the two ends of an arc a. Occasionally, the orientation of an arc is irrelevant to the discussion. In such instances, we refer to the arc as an edge of the directed graph. The number of arcs in D is denoted by $a(D)$. The vertices which dominate a vertex v are its **in-neighbors**, those which are dominated by the vertex are its **outneighbors**. These sets are denoted by $N_D^-(v)$ and $N_D^+(v)$, respectively.

The number of inward directed graph edges from a given graph vertex in a directed graph is called **indegree** and denoted by d^-. Similarly, the number of outgoing edges from a vertex in a directed graph is called **outderee** d^+. The degree $d(v)$ of a vertex v is defined by $d(v) = d^-(v) + d^+(v)$.

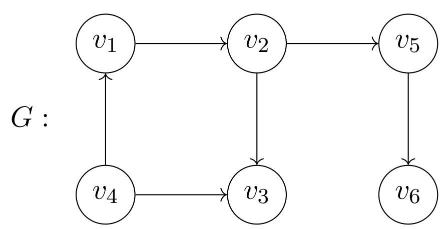


Figure 1.2.14: The digraph G

With any digraph D, we can associate a graph G on the same set of vertices simply by replacing each arc in D by an edge with the same ends. The new graph is the **underlying graph** of D, we denote it by $G(D)$. Any graph G can be regarded as a digraph, by replacing each of its edges by two oppositely oriented arcs with the same ends. This digraph is called the **associated digraph** of G, denoted $D(G)$. We can also obtain a digraph D from a graph G by replacing each edge by just one of the two possible arcs with the same ends. Such a digraph is then called an **orientation** of G. The orientation of G can be denoted by the symbol $\overrightarrow{G}$ to specify an orientation of G (even though a graph generally has many orientations). An orientation of a simple graph is referred to as an **oriented graph**. One particularly interesting instance is an orientation of a complete graph. Such an oriented graph is called a **tournament**, because it can be viewed as representing the results of a round-robin tournament, one in which each team plays every other team (and there are no ties).

Example 1.2.17. Figure 1.2.14 shows an orientation of a graph G from Example 1.2.2.

$\triangle$

A **stable set** in a digraph is a stable set in its underlying graph, that is, a set of pairwise nonadjacent vertices. If S is a maximal stable set in a graph G, then every vertex of $G - S$ is adjacent to some vertex of S. In the case of digraphs, it is natural to replace the notion of adjacency by the directed notion of dominance. This results in the concept of a kernel. A **kernel** in a digraph D is a stable set S of D such that each vertex of $D - S$ dominates some vertex of S.

1.2.9 Vertex colorings

A **vertex coloring** of a graph G refers to assigning colors to the vertices of G such that each vertex receives one color. A **proper vertex coloring** of a graph G is a function $c : V(G) \to \mathbb{N}$ such that for all $u, v \in V(G), c(u) \neq c(v)$ if $uv \in E(G)$. In proper vertex coloring no two adjacent vertices are assigned the same color. A k**-vertex-coloring** of a graph, or simply a k**-coloring**, is an assignment of k colors to its vertices. In a k-coloring, we may then assume that it is the colors $1, 2, \dots, k$ that are being used. A graph is k**-colorable** if it has a proper k-coloring. The **chromatic number** of G, $\chi(G)$, is defined as the least positive integer k such that G has a proper k-coloring for all $v \in V(G)$. If $\chi(G) = k$, then there exists a k-coloring of G, but not a $(k-1)$-coloring of G. A graph is k-colorable if and only if $\chi(G) \leq k$. Clearly, every graph of order n is n-colorable. For every graph G of order n, it holds that $1 \leq \chi(G) \leq n$. Surely, $\chi(K_n) = n$. If graph G is of order n and is not complete, then assigning the color 1 to two nonadjacent vertices of G and distinct colors to the remaining $n-2$ vertices of G gives us an $(n-1)$-coloring of G. A graph G of order n has chromatic number n if and only if G is complete. For a nonempty graph G to have chromatic number 2, there must be some partition of $V(G)$ into two independent subsets V_1, the vertices of G colored 1, and V_2, the vertices of G colored 2. Since every edge of G must join a vertex of V_1 and a vertex of V_2, the graph G is bipartite.

Theorem 1.2.1. *A nonempty graph G has chromatic number 2 if and only if G is bipartite.*

As a consequence of Theorem 1.2.1, we have that the chromatic number of a graph G is at least 3 if and only if G has an odd cycle.

Heuristic 1.2.1. *The Greedy coloring Heuristic:*
Input: a graph G
Output: a coloring of G
1. Arrange the vertices of G in a linear order: $v_1, v_2, \ldots, v_n$.
2. Color the vertices one by one in this order, assigning to v_i the smallest positive integer not yet assigned to one of its already-colored neighbors.

Using the Greedy coloring algorithm it can be easily shown that for every graph G it holds that $\chi(G) \leq 1 + \Delta(G)$, that is, the number of colors used by the greedy heuristic is never greater than $\Delta + 1$, regardless of the order in which the vertices are presented. When a vertex v is about to be colored, the number of its neighbors already colored is clearly no greater than its degree $d(v)$, and this is no greater than the maximum degree, Δ. Thus one of the colors $1, 2, \ldots, \Delta$ will certainly be available for v. We conclude that, for any graph G, $\chi(G) \leq \Delta(G) + 1$.

Theorem 1.2.2. *If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.*

A **clique** is a subset of vertices of an undirected graph such that every two distinct vertices in the clique are adjacent. That is, a clique of a graph G is an induced subgraph of G that is complete. A **maximal clique** is a clique that cannot be extended by including one more adjacent vertex, that is, a clique which does not exist exclusively within the vertex set of a larger clique. A **maximum clique** of a graph G is a clique, such that there is no clique with more vertices.

Definition 1.2.1. *The clique number, $\omega(G)$, of a graph G is the number of vertices in a maximum clique in G.*

Any clique of size $w(G)$ in G requires at least $w(G)$ different colors to color its vertices properly. Therefore, the chromatic number must be at least as large as the size of the largest clique, $\chi(G) \geq \omega(G)$. The following result is then an immediate consequence of Theorem 1.2.2.

Corollary 1.2.1. *For every graph G, $\chi(G) \geq \omega(G)$.*

1.2.10 Edge colorings

An **edge coloring** of a graph G is an assignment of colors to the edges of G, with each edge receiving one color. If adjacent edges are given different colors, the edge coloring is called **proper edge coloring**. Unless otherwise specified, any reference to edge coloring in this discussion will imply a proper edge coloring. Since a proper edge coloring of a nonempty graph G is equivalent to a proper vertex coloring of its line graph $L(G)$, studying edge colorings is essentially the same as studying vertex colorings of line graphs. However, because investigating vertex colorings of line graphs offers no clear advantage over directly examining edge colorings, we will focus exclusively on edge colorings in our study. An edge coloring that uses colors from a set of k colors is a **k-edge coloring**. A k-edge coloring of a graph G can be described as a function $c : E(G) \to \{1, 2, \ldots, k\}$ such that for every two adjacent edges e and f in G, it holds that $c(e) \neq c(f)$. A graph G is **k-edge colorable** if there exists a k-edge coloring of G. As in the case of vertex colorings, we are often interested in edge colorings using a minimum number of colors. The **chromatic index** or **edge chromatic number** $\chi'(G)$ of a graph G is the minimum positive integer k for which G is k-edge colorable. For every nonempty graph G it holds that $\chi'(G) = \chi(L(G))$. If a graph G is k-edge colorable for some positive integer k, then $\chi'(G) \leq k$. A graph G is said to belong to or is of **Class one** if $\chi'(G) = \Delta(G)$ and is of **Class two** if $\chi'(G) = 1 + \Delta(G)$.

Theorem 1.2.3 (König's Theorem)**.** *Every nonempty bipartite graph is of Class one.*

Although $\Delta(G)$ is an obvious lower bound for the chromatic index of a nonempty graph G, there are some examples that demonstrate that $\chi'(G)$ can be greater than $\Delta(G)$. In 1964, the Russian graph theorist Vadim G. Vizing [21] established an important upper bound for the chromatic index of a graph. Vizing's theorem is considered a major result in the study of edge colorings. It was independently rediscovered by Ram Prakash Gupta in 1966 [11].

Theorem 1.2.4 (Vizing's theorem)**.** *For every nonempty graph G,*

$$\chi'(G) \leq 1 + \Delta(G).$$

Chapter 2

List colorings

Over the last few decades, there has been a rising curiosity surrounding the colorings of graphs, wherein each vertex's color is to be picked from a designated list of permissible colors.

Definition 2.0.1. *Let G be a graph and let L be a function which assigns to each vertex $v \in V(G)$ a list $L(v)$, a set of positive integers. A proper coloring $c : V(G) \to \mathbb{N}$ such that $c(v) \in L(v)$ for all $v \in V$ is called a list coloring of G with respect to L or an $L - coloring$. In that case, we say that G is $L - colorable$ or $L - choosable$.*

A list coloring is also referred to as a $choice function$.

Definition 2.0.2. *Let G be a graph, k a positive integer and L a function such that $|L(v)| = k$ for all $v \in V(G)$. If the graph G has a proper list coloring, then G is k-choosable or k-list colorable and $\chi_L(G)$, the choice number or list chromatic number, is defined as the minimum k such that G has a proper list coloring for all lists assigned to the vertices of G.*

Note that k-choosable implies k-colorable, but not on the contrary as we will see in some future examples. A k-coloring is a special case of k-list-coloring, that is, if for all $v \in V(G)$, a list $L(v) = \{1, 2, \ldots, k\}$, then an L-coloring is simply a k-coloring. For example, if G is a bipartite graph and $L(v) = \{1, 2\}$ for all vertices $v \in V$, then G has the L-coloring which assigns color 1 to all vertices in one part and color 2 to all vertices in the other part of G. Every graph is clearly n-list colorable.

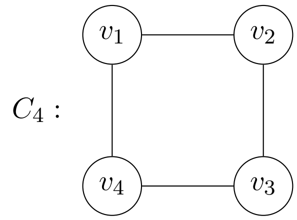


Figure 2.0.1: The graph C_4 is 2-list-colorable

Suppose that G is a graph with $\Delta(G) = \Delta$. If we let $L(v) = \{1, 2, \ldots, \Delta, 1+\Delta\}$ for each vertex v of G, then for these color lists there is a list coloring of G. Indeed, if $V(G) = v_1, v_2, \ldots, v_n$ and $\mathfrak{L} = \{L(v_i) : 1 \leq i \leq n\}$ is a collection of color lists for G where each set $L(v_i)$ consists of any $1 + \Delta$ colors, then a greedy coloring of G produces a proper coloring and so G is $\mathfrak{L}$-choosable. Therefore, $\chi_L(G) \leq 1 + \Delta(G)$. Recall that for graph G the list chromatic number is the minimum positive integer k such that G is k-choosable, therefore it holds that chromatic number is smaller or equal to list chromatic number, $\chi(G) \leq \chi_L(G)$. Summarizing these observations, we have the next theorem:

Theorem 2.0.1. *For every graph G,*

$$\chi(G) \leq \chi_L(G) \leq 1 + \Delta(G).$$

Let us now consider some examples.

Example 2.0.1. For graph C_4 it is $\chi(C_4) = 2$ and $\chi_L(C_4) \geq 2$. Suppose that for cycle C_4, shown in Figure 2.0.1, we are given any four color lists $L(v_i), 1 \leq i \leq 4$, such that $|L(v_i)| = 2$. Let $L(v_1) = \{a, b\}$. We consider three cases depending on whether color a is in lists $L(v_2)$ and $L(v_4)$.

- Case 1. The color a is in both lists, $a \in L(v_2) \cap L(v_4)$. We assign color b to v_1 and color a to v_2 and v_4. Then, we will have at least one color available in $L(v_3)$ that is different from color a and we can assign that color to the vertex v_3. This way we get a list coloring of C_4 for the given collection of lists.

- Case 2. The color a is in exactly one of the lists $L(v_2)$ and $L(v_4)$. Let, without loss of generality, $a \in L(v_2) \setminus L(v_4)$. We consider cases depending of whether intersection of $L(v_2)$ and $L(v_4)$ is empty or not.

If there exists some color $x \in L(v_2) \cap L(v_4)$, then assign v_2 and v_4 that color x and assign v_1 color a. There exists at least one color in $L(v_3)$ different from x and assign v_3 that color. For this collection of lists we showed that there is a list coloring of C_4.

If there does not exist some color belonging to both $L(v_2)$ and $L(v_4)$, we consider subcases whether color a is in list $L(v_3)$ or not. If $a \in L(v_3)$, then assign color a to both v_1 and v_3. There exists some other color available for vertices v_2 and v_4. If $a \notin L(v_3)$, then assign v_1 the color a, assign v_2 the color $y \in L(v_2) \backslash \{a\}$, assign v_3 any color $z \in L(v_3) \backslash \{y\}$ and assign v_4 any color in $L(v_4)$ that is different from color z to obtain a list coloring of cycle C_4.

- Case 3. The color a is not in any of these two lists, $a \notin L(v_2) \cup L(v_4)$. This means that there is an available color from $L(v_2)$ and $L(v_4)$ to assign to v_2 and v_4, respectively, and we found the list coloring of cycle C_4 in the last case too.

$\triangle$

We will show that for every even integer $n \geq 4$ it holds that $\chi_L(C_n) = 2$. First, let us show that $\chi_L(T) = 2$ for every nontrivial tree T. This will follow as a corollary of the theorem given below.

Theorem 2.0.2. *Every tree is 2-choosable. Furthermore, for every tree T, for a vertex u of T and for a collection $\mathfrak{L} = \{L(v) : v \in V(T)\}$, where each $L(u)$ is a color list of size 2 and $a \in L(u)$, there exists an $\mathfrak{L}$-list-coloring of T such that u is assigned color a.*

Proof. We will provide proof it by induction on number of vertices of tree T.

For a tree of order 1 or 2 the result is obvious. Assume that the statement holds for all trees of order k, $k \geq 2$. We will now prove it for the tree T of order $k+1$. Let

$$\mathfrak{L} = \{L(v) : v \in V(T)\}$$

be a collection of color lists of size 2. Let u be a vertex of G and suppose that it is colored a that is $a \in L(u)$. Let x be a leaf of tree T such that $x \neq u$ and let

$$\mathfrak{L}' = \{L(v) : v \in V(T-x)\}.$$

Let y be a neighbor of x in T. By the induction hypothesis, since tree $T-x$ is of order k, there exists an $\mathfrak{L}'$-list coloring c' of that tree in which u

is colored a. Let color $b \in L(x)$ such that $b \neq c'(y)$. If we define coloring c by

$$c(v) = \begin{cases} b & \text{if } v = x \\ c'(v) & \text{if } v \neq x \end{cases}$$

then c is an $\mathfrak{L}$-list coloring of tree T in which vertex u is colored a. ■

Corollary 2.0.1. *For every nontrivial tree T, $\chi_L(T) = 2$.*

We have a similar proof for showing that even cycles are also 2-list-colorable.

Theorem 2.0.3. *Every even cycle is 2-choosable.*

Proof. We will provide proof by induction on number of vertices of even cycle C.

By example **??** it is already proved that C_4 is 2-choosable. Let now C_n be an n-cycle, where n is even number, $n \geq 6$. Suppose that $C_n = v_1 v_2 \dots v_n v_1$.

Let

$$\mathfrak{L} = \{L(v_i) : 1 \leq i \leq n\}$$

be a collection of color lists of size 2 for vertices $v_i \in V(G)$. We will show that even cycle C_n is $\mathfrak{L}$-list-colorable. Consider two cases.

- Case 1. *The color lists in $\mathfrak{L}$ are all the same*, let us say $L(v_i) = \{1, 2\}$ for $1 \leq i \leq n$. If we assign e.g. color 1 to v_i for odd i and the color 2 to v_i for even i, then C_n is $\mathfrak{L}$-list-colorable.

- Case 2. *The color lists in $\mathfrak{L}$ are not all the same*. There are some adjacent vertices $v_i, v_{i+1} \in V(G)$ such that it holds $L(v_i) \neq L(v_{i+1})$. Therefore, there exists some color $a \in L(v_{i+1}) \setminus L(v_i)$. The graph $C_n - v_i$ is a path of length $n - 1$. Let

$$\mathfrak{L}' = \{L(v) : v \in V(C_n - v_i)\}.$$

By Theorem 2.0.2 there exists an $\mathfrak{L}'$-list-coloring c' of $C_n - v_i$ in which $c'(v_{i+1}) = a$. Let $b \in L(v_i)$ such that $b \neq c'(v_{i-1})$. If we define the coloring c as

$$c(v) = \begin{cases} b & \text{if } v = v_i \\ c'(v) & \text{if } v \neq v_i \end{cases}$$

then c is an $\mathfrak{L}$-list-coloring of C_n.

■

Corollary 2.0.2. *For every even integer $n \geq 4$, $\chi_L(C_n) = 2$.*

Since the chromatic number of every odd cycle is 3, the list chromatic number of every odd cycle must be at least 3. From Theorem 2.0.1 it follows that the upper bound of the list chromatic number of every odd cycle is $1 + \Delta = 3$. Combining this with previous conclusion we get that every odd cycle is 3-choosable.

We have shown that all nontrivial trees and even cycles are 2-choosable. Both of these are classes of bipartite graphs. But not every bipartite graph is 2-choosable. Let us demonstrate this by giving a counter example.

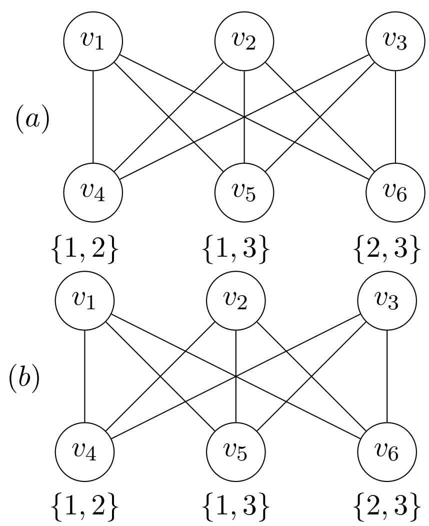


Figure 2.0.2: The graph $K_{3,3}$ is 3-choosable

Example 2.0.2. Consider $\chi_L(K_{3,3})$, where $K_{3,3}$ is shown in Figure 2.0.2 (a). By Theorem 2.0.1, $\chi_L(K_{3,3}) \leq 1 + \Delta(K_{3,3}) = 4$. Actually, a stronger statement holds, that $\chi_L(K_{3,3}) \leq 3$, as will be showed next.

Let there be given lists of size 3, $L(v_i), 1 \leq i \leq 6$. We consider two cases.

- Case 1. Some color occurs in two or more of the lists $L(v_1), L(v_2), L(v_3)$ or in two or more of the other three lists $L(v_4), L(v_5), L(v_6)$. Without loss of generality, let us assume color a occurs in both $L(v_1)$ and $L(v_2)$. Then assign vertices v_1 and v_2 the color a and assign v_3 any color in list $L(v_3)$. Then, there is an available color in $L(v_i)$ for $v_i (4 \leq i \leq 6)$.

- Case 2. The lists $L(v_1), L(v_2), L(v_3)$ are pairwise disjoint as are the lists $L(v_4), L(v_5), L(v_6)$. Let $a_1 \in L(v_1)$ and $a_2 \in L(v_2)$. We have two subcases.
  - Subcase 2.1. Both a_1 and a_2 are in none of the lists $L(v_4), L(v_5), L(v_6)$, then let a_3 be any color in $L(v_3)$. Then there is an available color for each of the vertices v_4, v_5, v_6 to obtain a proper coloring of $K_{3,3}$.
  - Subcase 2.2. Exactly one of the lists $L(v_4), L(v_5), L(v_6)$ contains both a_1 and a_2, then select a color $a_3 \in L(v_3)$ so that none of $L(v_4), L(v_5), L(v_6)$ contains all of a_1, a_2, a_3. By assigning v_3 the color a_3, we see that there is an available color for each of v_4, v_5 and v_6.

Therefore, we have proved that $\chi_L(K_{3,3}) \leq 3$. Actually, it can be shown that equality holds i.e, $\chi_L(K_{3,3}) = 3$.

Consider the sets $L(v_i), 1 \leq i \leq 6$, shown in Figure 2.0.2 (b). Assume, without loss of generality, that v_1 is colored 1. Then v_4 must be colored 2 and v_5 must be colored 3. Whichever color is chosen for v_3 is the same color as that of either v_4 or v_5. This produces a contradiction. Hence, $K_{3,3}$ is not 2-choosable and so we get that $\chi_L(K_{3,3}) = 3$. $\triangle$

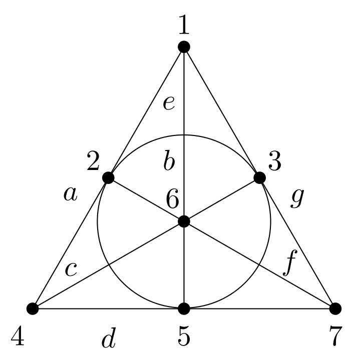


Figure 2.0.3: The Fano plane

Example 2.0.3. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{P} = \{a, b, c, d, e, f, g\}$, where

- $a = \{1, 2, 4\}$,
- $b = \{2, 3, 5\}$,
- $c = \{3, 4, 6\}$,
- $d = \{4, 5, 7\}$,
- $e = \{5, 6, 1\}$,
- $f = \{6, 7, 2\}$.
- $g = \{7, 1, 3\}$,

Then $(X, \mathcal{P})$ is a finite projective plane, called Fano plane (see Figure 2.0.3).

Note that in Figure 2.0.3, the Fano plane is represented by seven lines, where six are line segments and one is circle. However, formally, each of these lines is simply a set of three points. Drawings such this one in Figure 2.0.3 can sometimes be useful for guiding our intuition. Nonetheless, formal proofs should never rely on such pictures; instead, they should rely solely on the definition of a finite projective plane or on results proven about them.

Example 2.0.4. Complete graph $K_{7,7}$ showed in the Figure 2.0.4, using Fano plane for coming up with the combination of colors, is not 3-choosable.

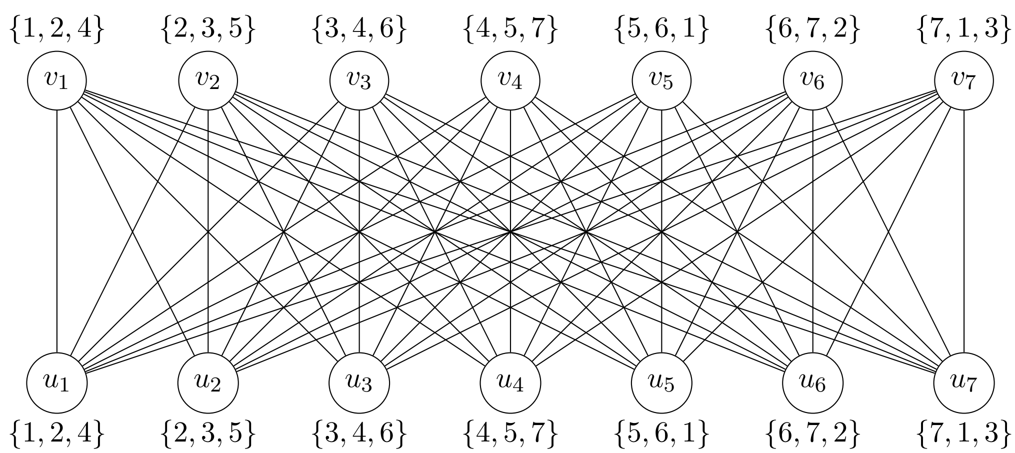


Figure 2.0.4: The graph $K_{7,7}$ is not 3-choosable

The graph $G = K_{3,3}$ in Figure 2.0.2 shows that it is possible for $\chi_L(G) > \chi(G)$. In fact, $\chi_L(G)$ can be considerably larger than $\chi(G)$ as will be proven in the theorem below.

Theorem 2.0.4. *If r and k are positive integers such that $r \geq \binom{2k-1}{k}$, then*

$$\chi_L(K_{r,r}) \geq k + 1.$$

Proof. Assume on the contrary, that $\chi_L(K_{r,r}) \leq k$. Then, $K_{r,r}$ is k-choosable.

Let U and W be partitive sets of $K_{r,r}$, such that $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_r\}$.

Let $S = \{1, 2, \ldots, 2k-1\}$. There are $\binom{2k-1}{k}$ distinct k-element subsets of S. Assign these color lists to $\binom{2k-1}{k}$ vertices of U and to $\binom{2k-1}{k}$ vertices of W. The rest of the vertices of both U and W are assigned any of the k-element subset of S. For $i = 1, 2, \ldots, r$ choose a color $a_i \in L(u_i)$ and let $T = \{a_i : 1 \leq i \leq r\}$. Consider two cases depending on the size of T.

- Case 1. $|T| \leq k-1$. Then there exists a k-element subset S' of S such that it is disjoint from T. Since $L(u_j) = S'$ for some j, $1 \leq j \leq r$, this is a contradiction.

- Case 2. $|T| \geq k$. Therefore, there exists a k-element subset T' of T. Accordingly, $L(w_j) = T'$ for some j, $1 \leq j \leq r$. Whichever color from $L(w_j)$ is assigned to w_j, this color has already been assigned to some vertex u_i. We get that u_i and w_j have been assigned the same color and since $u_i w_j$ is an edge in $K_{r,r}$ this is a contradiction.

■

For positive integers r and k, such that $r \geq \binom{2k-1}{k}$, Theorem 2.0.4 ensures a lower bound on $\chi_L(K_{r,r})$ for r sufficiently large with respect to k. There is also an upper bound for the $\chi_L(K_{r,r})$ for all $r \geq 3$, discovered by Noga Alon in 1992 given in the following theorem.

Theorem 2.0.5. *For every integer $r \geq 3$,*

$$\chi_L(K_{r,r}) \leq \lceil 2 \log_2 r \rceil.$$

Bounds on the list chromatic number of certain graphs can be found by means of kernels too. Even though the kernel is a notion which concerns directed graphs and the list chromatic number concerns undirected graphs, the following theorem gives a link between kernels and list colorings.

Theorem 2.0.6. *Let G be a graph and let D be an orientation of G each of whose induced subdigraphs has a kernel. Let $L(v)$ be an arbitrary list of at least $d_D^+(v) + 1$ colors, $v \in V(G)$. Then G has an L-coloring.*

Proof. By induction on n.

For $n = 1$ the statement is obvious. Assume the statement holds for n and we will show it for $n + 1$.

Let V_1 be the set of vertices of D whose lists include color 1. Without loss of generality, assume $V_1 \neq \emptyset$ by renaming colors if it is necessary. By assumption, $D[V_1]$ has a kernel S_1. Color vertices of the kernel S_1 with color 1. Let $G' := G - S_1, D' := D - S_1$ and $L'(v) := L(v) \backslash \{1\}$. For any vertex v of D' whose list did not contain color 1,

$$|L'(v)| = |L(v)| \geq d_D^+(v) + 1 \geq d_{D'}^+(v) + 1$$

and for any vertex v of D' whose list contains color 1,

$$|L'(v)| \geq |L(v)| - 1 \geq d_D^+(v) \geq d_{D'}^+ + 1. \tag{2.1}$$

In D the vertex v dominates some vertex of the kernel S_1, so its outdegree in D' is smaller than in D, hence the inequality (2.1) holds. By induction G' has an L'-coloring. When combined with the coloring of kernel S_1, we get an L-coloring of graph G.

■

To illustrate Theorem 2.0.6, consider the case where D is an acyclic orientation of G. Since every acyclic digraph has a kernel, D satisfies the hypothesis of the theorem. Clearly, $d_D^+(v) \leq \Delta^+(D) \leq \Delta(G)$. Hence, Theorem 2.0.6 shows that G has a list coloring whenever each list is comprised of $\Delta + 1$ colors.

A similar approach can be applied to list colorings of interval graphs. Woodall in 2001 showed that every interval graph G has an acyclic orientation D with $\Delta^+ \leq \omega - 1$.

Corollary 2.0.3. *Every interval graph G has list chromatic number ω.*

The following theorem shows that, in terms of its relationship to other graph invariants, the choice number differs fundamentally from the chromatic number. As mentioned before, there are 2-chromatic graphs of arbitrarily large minimum degree, e.g. the graph $K_{n,n}$. The choice number, however, will be forced to up by large values of invariants.

Theorem 2.0.7 (Alon [6]). *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any integer k, all graphs G with an average degree $d(G) \geq f(k)$ satisfy $\chi(G) \geq k$.*

2.1 2-Choosable Graphs

Within this section, we provide a characterization of all 2-choosable graphs, as outlined in [1], originally from [8]. Before delving into that, however, it is necessary to establish a few definitions. Recall that graph G is k-choosable if there is a list of length k on each vertex and from any such set of lists the graph G may be properly colored.

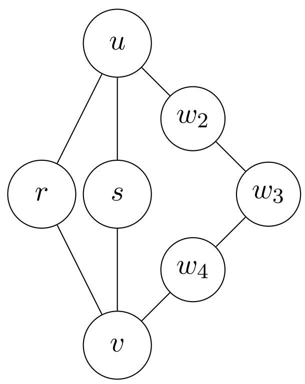


Figure 2.1.1: The graph $\theta_{2,2,4}$

Let us define a $\theta_{a,b,c}$ graph to be a graph with two distinct vertices, u and v, with three vertex-disjoint paths between them having lengths a, b, and c, respectively. A graph $\theta_{a,b,c}$ is identified by the lengths of these three paths. See Figure 2.1.1 for an example of graph $\theta_{2,2,4}$, where these three paths between vertices u and v are of lengths 2, 2 and 4, respectively.

Proposition 2.1.1. If a core of a graph G if 2-choosable, then graph G is 2-choosable.

Proof. Recall that the core of a graph G is actually graph G with all vertices of degree one recursively removed. These vertices of degree one may always be colored from lists of length two, since they only have one neighbor to conflict with a coloring, leaving at least one color available for each of these vertices. Therefore, if a core of G is 2-choosable, we can color the removed vertices with available colors from lists of length 2 and we get the proper coloring of graph G. ■

We will now prove the main theorem of the section.

Theorem 2.1.1 (Rubin, [8]). *A graph G is 2-choosable if and only if the core of G is K_1, an even cycle or of the form $\theta_{2,2,2k}$, where k is a positive integer.*

Proof. $Sufficiency$

Assume, first, that the core of graph G is either K_1, an even cycle or of the form $\theta_{2,2,2k}$. We will prove that the core of G is 2-choosable. Then, by Proposition 2.1.1 it follows that G is 2-choosable.

If G is K_1, since K_1 has only one vertex, it is obviously 2-choosable. In the second case, a graph formed from K_2 with a parallel edge is clearly 2-choosable, so we will assume that core of G is either an even cycle of length at least 4 or a graph of form $\theta_{2,2,2k}$.

Even cycles are subgraphs of $\theta_{2,2,2k}$ graphs, so we will prove the statement for graphs of form $\theta_{2,2,2k}$ and it will follow for even cycles or we can just recall Theorem 2.0.3 where we already proved 2-choosability of even cycles.

Let us label vertices in the $\theta_{2,2,2k}$ as in Figure 2.1.1. On the two paths of length two, between u and v, let us denote vertices by r and s, and denote the vertices on the path of length $2k$ with w_i, $1 \leq i \leq 2k+1$. Vertices u and v correspond with w_1 and w_{2k+1} respectively.

We consider two cases, depending on if all lists on w_i, $1 \leq i \leq 2k+1$, are the same or not.

- Case 1. *The lists are all the same.* Let $L(w_i) = \{a, b\}$. Then choose, without loss of generality, color a for odd vertices and color b for even. Note that both $u = w_1$ and $v = w_{2k+1}$ will be colored a, and that both lists for r and s must contain a color different from a and we can color r and s as well to get a proper coloring of $\theta_{2,2,2k}$.

- Case 2. *The lists are not the same.* Then there exist some j such that $w_j w_{j+1}$ is an edge in G and $L(w_j) \neq L(w_{j+1})$. For w_j choose a color a_j that is in $L(w_j)$, but not in $L(w_{j+1})$. For w_{j-1} choose a color a_{j-1} that is in $L(w_{j-1})$, but is different from color a_j. We continue this process in recursion until we have colored $u = w_1$ with color a_1.

  For vertex $v = w_{2k+1}$ suppose that the list of colors is given by $L(v) = \{r_1, s_1\}$.

  If $L(r) \neq \{a_1, r_1\}$ or $L(s) \neq \{a_1, s_1\}$, then we can choose colors for r, s and w_{2k+1} and just continue the coloring from w_{2k+1} back to w_{j+1}.

  If $L(r) = \{a_1, r_1\}$ and $L(s) = \{a_1, s_1\}$, then we are forced to choose color r_1 for r and s_1 for s, so we cannot color w_{2k+1}, as shown in Figure

2.1.2. We will then return to the vertex w_{j+1} to begin again. For w_{j+1} we now choose color a_{j+1} that is in $L(w_{j+1})$, but not in $L(w_{j+2})$.

We continue this process for vertex w_{j+2} by choosing color a_{j+2} that is in $L(w_{j+2})$, but is different from a_{j+1} and similarly for the rest of the vertices till we get to w_{2k+1}. Since $L(w_{2k+1}) = \{r_1, s_1\}$ and $L(r) = \{a_1, r_1\}$ we get that $a_1 \neq r_1$. Hence, we choose one of r_1 or s_1 for vertex w_{2k+1} and color a_1 for both vertices r and s. Then, a color different from a_1 is left in $L(w_1)$ and we can continue the coloring.

We have showed that graphs of form $\theta_{2,2,2k}$ are 2-choosable and so are even cycles of order ≥ 4.

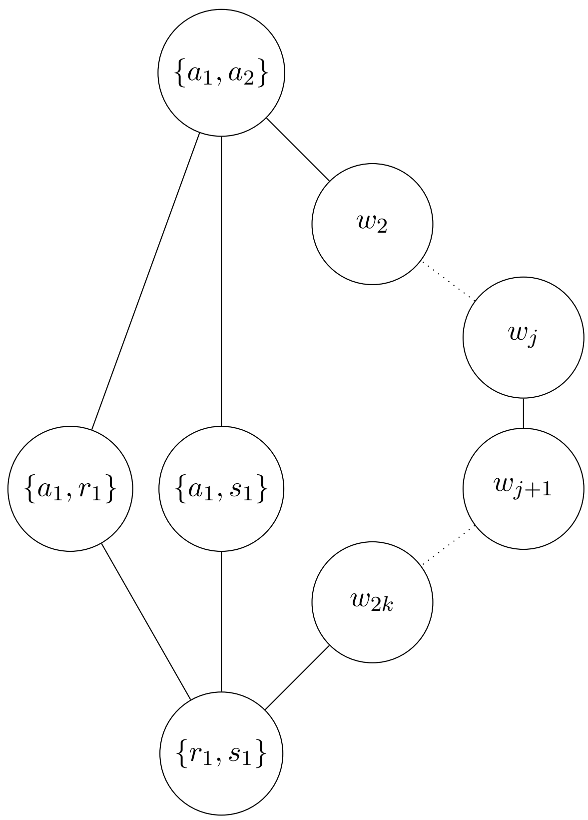


Figure 2.1.2: Not all w_i have the same lists

Necessity

Assume now that G is 2-choosable. We will show that the core of G must be K_1, an even cycle or $\theta_{2,2,2k}$.

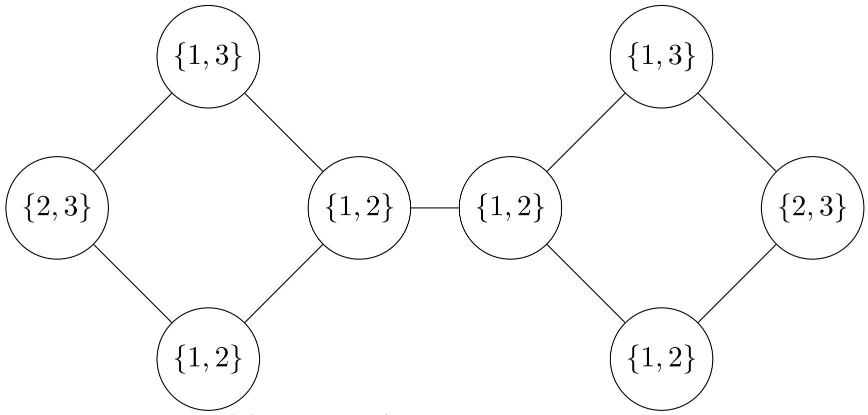
Suppose on the contrary, that G is a 2-choosable graph whose core is not K_1, an even cycle or $\theta_{2,2,2k}$. Since vertices with degree one are always 2-choosable, we will assume that they have been removed from G and will just look at the core of G.

If G does not contain a cycle, then the core of G is K_1 which is impossible by assumption, hence G contains a cycle and let us denote it by C'. If C' is an odd cycle, then $\chi(C') = 3$. Since $\chi_L(G) \geq \chi(G)$ and we have assumed that $\chi(G) = 2$ it follows that G cannot contain an odd cycle. We have also assumed that G is not an even cycle, so if C is the shortest cycle in G, then there exists an edge that is in G and not in C. Since the degree of G is 2, this edge lies on another cycle or on a path connecting cycles. If C^* is another cycle in G, then, since G is a connected graph, C and C^* are connected by a path or they share a vertices.

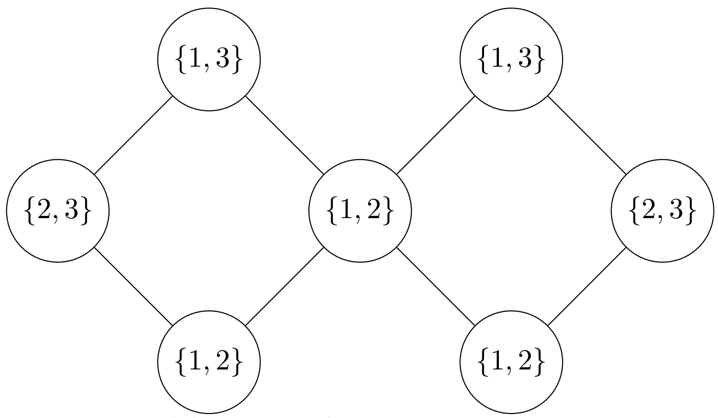
Assume first that C and C^* share at most one vertex. We then remove vertex x from G and merge the vertices that were adjacent to x, also remove any parallel edges, that are produced by the process of vertex reduction, since parallel edges do not affect vertex coloring. We repeat this process of vertex reduction until we are left with either one of Figure 2.1.3(a) or 2.1.3(b).

Note that neither one of graphs on Figures 2.1.3(a) or 2.1.3(b). is not 2-choosable with the given lists.

Recall that the union of graphs G_1 and G_2 is a graph $G = G_1 + G_2$ which has vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1)$ $\cup$ $E(G_2)$.



(a) C and C* share no vertices



(b) C and C* share 1 vertex

Figure 2.1.3: Cycles joined by at most 1 vertex

Assume now that C and C^* share at least two vertices. This means that there exists an edge-disjoint path from C that is connecting two distinct vertices in C. Denote by P the shortest such path. $C \cup P$ can be of the form $\theta_{2,2,2k}$ or not.

If $C \cup P$ is not of the form $\theta_{2,2,2k}$, then it must be $\theta_{a,b,c}$, $a \neq 2, b \neq 2$. We can reduce $C \cup P$ to graph in the Figure 2.1.4. Note that this graph is also not 2-choosable with the given list.

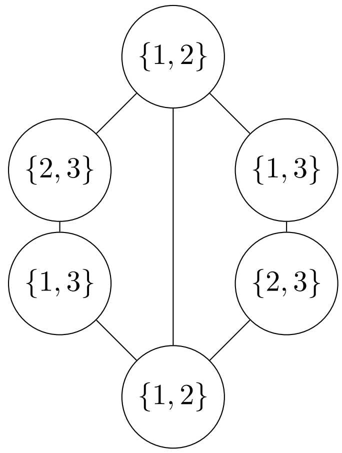


Figure 2.1.4: A $\theta_{a,b,c}$ graph with $a, b \neq 2$

If $C \cup P$ is of the form $\theta_{2,2,2k}$ and since C is the shortest cycle in G, then C is a 4-cycle. Let us label the vertices in C as in the Figure 2.1.5. Since we have assumed that G is not of form $\theta_{2,2,2k}$, then we must have another shortest path P^* such that P^* is edge-disjoint from $C \cup P$ and connects two distinct vertices in $C \cup P$. We have six possible cases now:

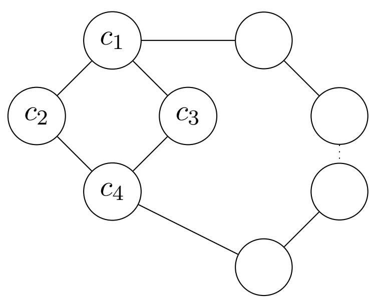


Figure 2.1.5: $C \cup P$

- Case 1. The endpoints of P^* are interior vertices of P. Then, we get two edge-disjoint cycles which are connected with a path and we can reduce them to Figure 2.1.3(a).
- Case 2. One endpoint of P^* is c_1 and the other one is an interior vertex of P. Then, we have two edge-disjoint cycles which share a vertex and we can reduce them to Figure 2.1.3(b).
- Case 3. One endpoint of P^* is c_3 and the other one is an interior

vertex u of P. Then, the edge c_1c_3, and paths $c_1c_2c_4c_3$ and the one from c_1 to u through P joined with a path from u to c_3 through P^* forms a $\theta_{a,b,c}$ graph with $a \neq 2$ and $b \neq 2$. Then, we can reduce this graph to Figure 2.1.4.

- Case 4. One endpoint of P^* is c_1 and the other one is c_3. Then, we get the same case as previous, Case 3, since we have that P^* and the paths $c_1c_2c_4c_3$ and c_1c_3 give us a $\theta_{a,b,c}$ graph with $a \neq 2$ and $b \neq 2$.

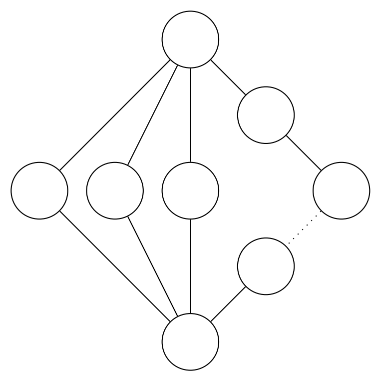


Figure 2.1.6: A θ graph with an extra 2-path

- Case 5. The endpoints of P^* are c_1 and c_4. If P has length two, then we get a graph which is of form shown in Figure 2.1.6. Then, we can reduce this graph to Figure 2.1.7 which is not 2-choosable. If the length of P is greater than two, then the length of P^* is also greater than two. Then, the edge c_1c_3 and the paths P^* and $c_1c_2c_4c_3$ form a $\theta_{a,b,c}$ graph such that $a \neq 2$ and $b \neq 2$ as we have had before in Cases 3 and 4.

- Case 6. The endpoints of P^* are c_2 and c_3. Then, we remove the edge c_1c_3. Note that the edge c_2c_4, P^* and the path from c_2 to c_4 through the edge c_2c_1 and P form a graph $\theta_{a,b,c}$ such that $a \neq 2$ and $b \neq 2$.

We will now prove that if G' is a reduction of G obtained by our method mentioned above and G' is not 2-choosable, then G is not 2-choosable.

Let G' be obtained by deleting vertex x from graph G. Undo the reduction by unmerging the vertices of G which were adjacent to x and adding x back into the graph. To all of these vertices assign the same list, {a,b}, that was on the merged vertex in G'. If we choose color a for x, then we must choose b for every vertex adjacent to x. Then, if this choice would have created a proper coloring of G it would have also created a proper coloring of G', which is a contradiction.

Hence, graph G is not 2-choosable, our reductions in Figures 2.1.3(a), 2.1.3(b) 2.1.4 and 2.1.7 show that the only graphs which are 2-choosable have cores that are either K_1, even cycles or of form $\theta_{2,2,2k}$.

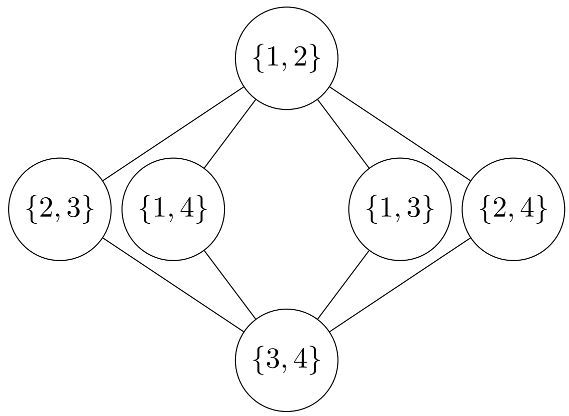


Figure 2.1.7: Reduction of a θ graph with an extra 2-path

■

2.2 Planar Graphs

2.2.1 The Five-Color Theorem

From an empirical standpoint, another phenomenon underscores the distinct nature of choice numbers in comparison to chromatic numbers: even when known bounds for the chromatic number can be applied to the choice number, the methodologies used in their proofs often diverge.

A notable illustration of this lies in the list version of the five-color theorem: the conjecture that any planar graph is 5-choosable, given by Vadim Vizing in 1976 and Paul Erdös, Arthur L. Rubin and Herbert Taylor in 1980. This conjecture persisted for nearly two decades until Carsten Thomassen, in 1994, formulated a very straightforward induction proof. Notably, this proof operates independently of the five-color theorem, resulting in a fundamentally disparate demonstration of its validity.

Recall that a near-triangulation is a plane graph all of whose inner faces are of degree three.

Theorem 2.2.1 (Thomassen [20]). *Let G be a near-triangulation whose outer face is bounded by a cycle C, and let x and y be two consecutive vertices of C. Suppose that $L : V \to 2^{\mathbb{N}}$ is an assignment of lists of colors to the vertices of G such that:*

1. $|L(x)| = |L(y)| = 1$, *where* $L(x) \neq L(y)$,
2. $|L(v)| \geq 3$, *for all* $v \in V(C) \backslash \{x, y\}$,
3. $|L(v)| \geq 5$, *for all* $v \in V(G) \backslash V(C)$.

Then, G is L-colorable.

Proof. By induction on number of vertices $|V(G)|$.

If $|V(G)| = 3$, then $G = C$ and the statement is certainly true. Let us assume that $|V(G)| \geq 3$. Denote by z and x' immediate predecessors of x on cycle C, respectively. We will consider two cases depending on if x' has only z and x as neighbors on C or not.

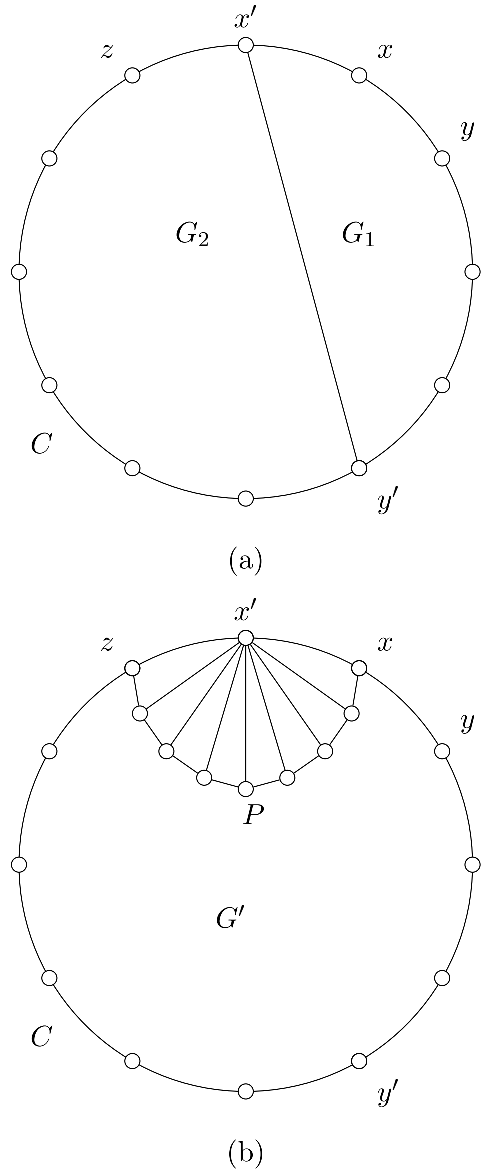


Figure 2.2.1

- Case 1. Assume x' has a neighbor on C other than x and z and denote it y' (see Figure 2.2.1 (a)). Let $C_1 = x'Cy'x'$ and $C_2 = x'y'Cx'$. C_1 and C_2 are two cycles on G and note that G is the union of near-triangulation G_1 consisting of cycle C_1 together with its interior and the near-triangulation of G_2 consisting of cycle C_2 together with its interior. Let

$$L_1(v) = \{L(v) : v \in V(G_1)\},$$

By induction hypothesis G_1 has an L_1-coloring c_1. Let L_2 be the function defined on $V(G_2)$ such that

$$L_2(x') = \{c_1(x')\}, L_2(y') = \{c_1(y')\}$$

and

$$L_2(v) = \{L(v) \text{ for } v \in V(G_2)\backslash\{x', y'\}.$$

By induction hypothesis G_2 has an L_2-coloring c_2. By the definition of L_2, we have that $c_1(x') = c_2(x')$ and $c_1(y') = c_2(y')$, remember that $x', y' \in G_1 \cap G_2$. Therefore, the function c defined by

$$c(v) = \begin{cases} c_1(v), v \in V(G_1) \\ c_2(v), v \in V(G_2)\backslash V(G_1) \end{cases}$$

is an L-coloring of G.

- Case 2. Assume x' has a neighbor on path xPz internally disjoint from C, as shown in Figure 2.2.1 (b). Let $G' = G - x'$, note that G' is a near-triangulation whose outer face is bounded by the cycle $C' := xCzP^{-1}x$. Let $\alpha, \beta \in L(x')\backslash L(x)$ such that $\alpha \neq \beta$. Let L' be a function on $V(G')$ defined by

$$L'(v) = L(v)\backslash\{\alpha, \beta\} \text{ for } v \in V(P)\backslash\{x, z\}$$

and

$$L'(v) = L(v)$$

for all other vertices v of G'.

By induction hypothesis, there exists an L'-coloring c' of G'. We have that either $\alpha \neq c'(z)$ or $\beta \neq c'(z)$. Let us assume, without loss of generality, that $\alpha \neq c'(z)$. If we extend the coloring c' by assigning color α to the vertex x', then we get an L-coloring c of graph G.

■

An immediate consequence of a Theorem 2.2.1 is the following strengthening of the Five-Color Theorem.

Corollary 2.2.1. *Every planar graph is 5-choosable.*

This stands as one of the more enlightening demonstrations of the Five-Color Theorem. Unfortunately, there is no equivalent list coloring analogue corresponding to the Four-Color Theorem that is currently known. In fact, Voigt's work in [24] unveiled examples of planar graphs which are not 4-list-colorable

Nevertheless, there remains the possibility that a suitable list coloring version of the Four-Color Theorem could present a more transparent and shorter proof of that theorem as well. (For instance, Kündgen and Ramamurthi in [16] have proposed the idea that every planar graph admits a list coloring when the available colors are grouped in pairs, and each list consists of two of these pairs.)

Voigt, in [23], also provided examples of triangle-free planar graphs that are not 3-list-colorable. These instances show the absence of an intuitive list coloring extension for the following 'three-color theorem' originally formulated by Grötzsch in [10].

Theorem 2.2.2 (Grötzsch's Theorem)**.** *Every triangle-free planar graph is* 3*-colorable.*

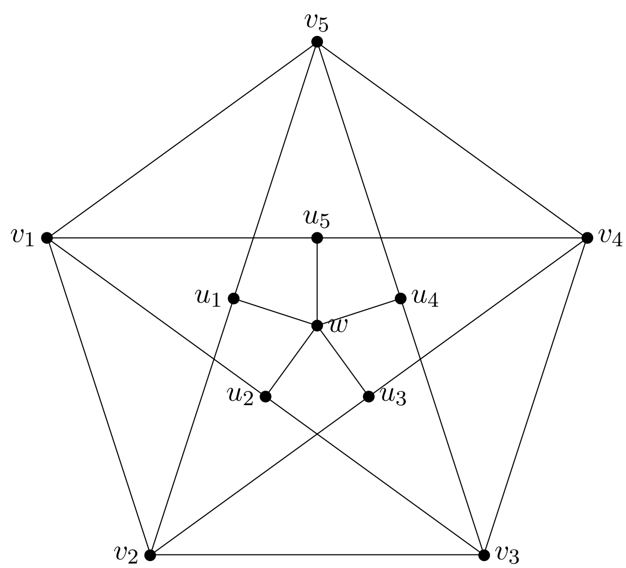


Figure 2.2.2: The Grötzsch graph: a 4-critical graph

However, an interesting discovery emerges: any planar graph with a girth of five is 3-list colorable. Thomassen established this outcome in 1994 through reasoning that is close to, yet more intricate than, the arguments he utilized to establish Theorem 2.2.1. This can arguably be seen as an extension of Grötzsch's Theorem in the field of list coloring, given that the latter can be relatively easily reduced to planar graphs with a girth of five.

Grötzsch's Theorem can also be demonstrated using a methodology quite similar to that of the Four-Color Theorem, although with notably simpler reasoning. The 4-chromatic Grötzsch graph (shown in Figure 2.2.2) shows that the scope of Grötzsch's Theorem does not cover non-planar graphs. Indeed, Grötzsch constructed this graph in [10] specifically for this purpose.

2.2.2 The Mirzakhani Graph

In 1993, Margit Voigt provided an example of a planar graph containing 238 vertices that is not 4-choosable. Subsequently, in 1996, Maryam Mirzakhani presented an even simpler example, specifically a planar graph with 63 vertices that is not 4-choosable. In the year 2014, Mirzakhani received the Fields Medal, a prestigious acknowledgment bestowed upon mathematicians under 40 years of age, often compared to the mathematical equivalent of a Nobel Prize. Notably, Mirzakhani was the first woman to attain this honor. Let us now describe the Mirzakhani graph, explaining its properties, and confirming its incapacity for being 4-choosable.

Let H be a planar graph of order 17 shown in the Figure 2.2.3 (a). For each vertex $v \in V(H)$, a list $L(v) \subseteq \{1, 2, 3, 4\}$, is assigned as in Figure 2.2.3 (b), note that $|L(v)| \in \{3, 4\}$. Actually, if $d_H(v) = 4$, then $L(v) = \{1, 2, 3, 4\}$ and if $d_H(v) \neq 4$, then we have that $|L(v)| = 3$. Let

$$\mathfrak{L} = \{L(v) : v \in V(H)\}.$$

we will prove that H is not $\mathfrak{L}$-choosable in the following theorem.

Theorem 2.2.3. *The planar graph shown in Figure 2.2.3 (a), with the set of colors, $\mathfrak{L}$, as in Figure 2.2.3 (b), is nor $\mathfrak{L}$-choosable.*

Proof. Assume to the contrary, that H is $\mathfrak{L}$-choosable.

Then, there exists a 4-coloring c of graph H, $c(v) \in L(v), v \in V(H)$. In graph H every vertex that has a degree 4 is adjacent to vertices assigned either two or three distinct colors, so it follows that two nonadjacent vertices, of those 4 neighbors of vertices of degree 4, are assigned the same color. We will prove that $c(x) = 1$ or $c(w) = 2$.

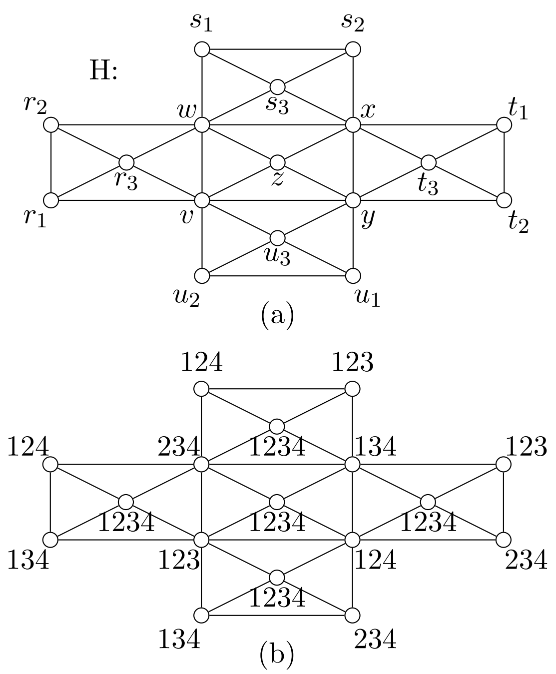


Figure 2.2.3: A planar graph of order 17

Assume to the contrary, that $c(x) \neq 1$ and $c(w) \neq 2$. Then $c(x), c(w) \in \{3.4\}$. Let us first consider the case if $c(x) = 3$ and $c(w) = 4$. Then we have that whether $c(s_1) = 3$ or $c(s_2) = 4$. Since $3 \notin L(s_1)$ and $4 \notin L(s_2)$, this case is not possible. Suppose now that it is $c(x) = 4$ and $c(w) = 3$. Then, it will have to be that either $c(v) = 4$ or $c(y) = 3$. Here, we have similar scenario as before, since $4 \notin L(v)$ and $3 \notin L(y)$, therefore this case is not possible either. Hence, we get that one of the following cases is possible, it is $c(x) = 1$ or $c(w) = 2$.

- Case 1. Let us assume first that $c(x) = 1$. Then the color of the vertices t_1, t_2 and y cannot be 1 and it follows that $c(t_1) = c(y) = 2$. Since the color of the vertices u_1, u_2 and v cannot be 2, it follows that $c(u_1) = c(v) = 3$. Hence, none of the vertices r_1, r_2 and w cannot be colored 3, so we get that $c(r_1) = c(w) = 4$. Now, we get that all 4 neighbors of z take distinct colors from set $\{1, 2, 3, 4\}$, i.e $c(x) = 1, c(y) = 2, c(v) = 3, c(w) = 4$, which is not possible, so $c(x)$ cannot be colored 1.
- Case 2. Let us now assume that $c(w) = 2$. By proceeding similarly

as in the first case, we get that $c(r_2) = c(v) = 1$. It follows that $c(u_2) = c(y) = 4$. Then we get that $c(t_2) = c(x) = 3$. Once again we get that all 4 neighbors of z have all 4 colors, i.e $c(v) = 1, c(w) = 2, c(x) = 3, c(y) = 4$ which is not possible, so $c(w)$ cannot be colored 2.

Hence, the graph H is not $\mathfrak{L}$-choosable for the given set L of color lists described in Figure 2.2.3 (b).

■

Following the description of the Mirzakhani graph, let H_1, H_2, H_3 and H_4 be four copies of of the graph H shown in Figure 2.2.3 (a). Replace by 5 the color i, for $1 \leq i \leq 4$, in the color list of every vertex of H_i in Figure 2.2.3 (b) and add color i to the color list of each vertex which does not have degree 4. The obtained graphs H_i,$1 \leq i \leq 4$, are shown in Figure 2.2.4.

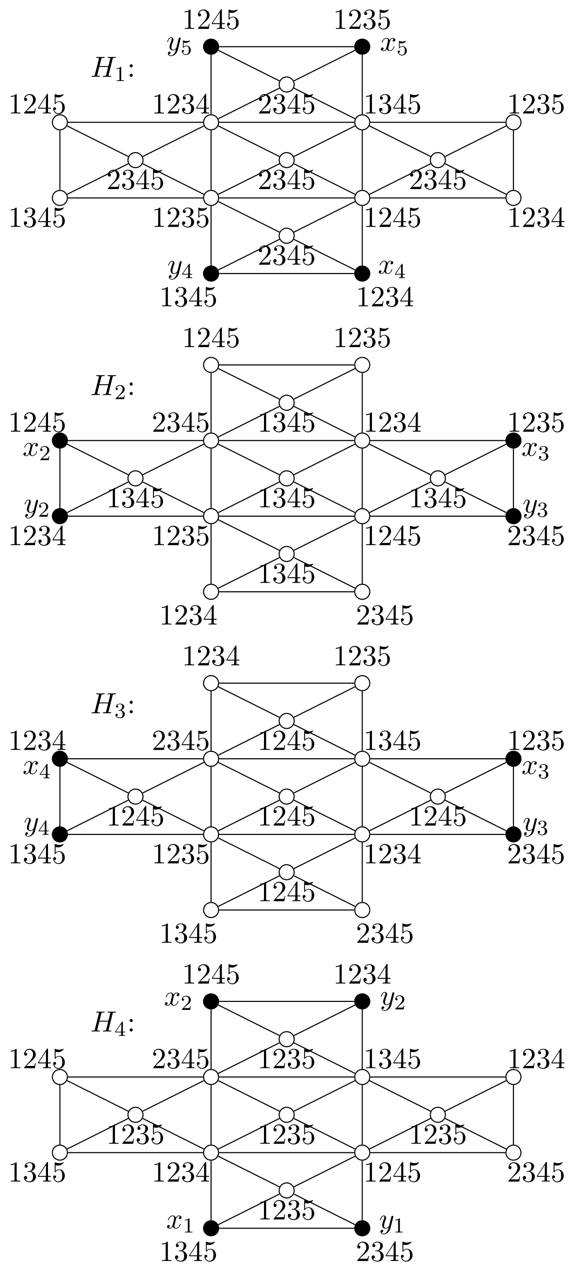


Figure 2.2.4: The graphs $H_i, 1 \leq i \leq 4$

The Mirzakhani graph G is a planar graph of order 63 constructed in the following way: from the previously defined graphs $H_i, 1 \leq i \leq 4$, in the Figure 2.2.4, by identifying two vertices labeled x_i and two vertices labeled y_i, $2 \leq i \leq 4$, and adding a new vertex, p, such that $L'(p) = \{1, 2, 3, 4\}$ and then joining p to each vertex of each copy of H_i of H whose degree is not equal 4. The Mirzakhani graph with the resulting color lists for each vertex is now shown in Figure 2.2.5. Let

$$\mathfrak{L}' = \{L'(v) : v \in V(G)\}.$$

We will show that the graph G is not $\mathfrak{L}'$-choosable.

Theorem 2.2.4. *The Mirzakhani graph is not* 4*-choosable.*

Proof. Let $L'(v)$ be the color list for each vertex $v \in V(G)$ shown in Figure 2.2.5 and let

$$\mathfrak{L}' = \{L'(v) : v \in V(G)\}.$$

we will prove that the graph G is not $\mathfrak{L}'$-choosable.

Assume on the contrary, that graph G is $\mathfrak{L}'$-choosable.

Then, there exists a coloring c' such that $c'(v) \in L'(v)$, $v \in V(G)$. Since we have that the graph H of Figure 2.2.3(a) is not $\mathfrak{L}$-choosable for the set $\mathfrak{L}$ in Figure 2.2.3(b), the only for G to be $\mathfrak{L}'$-choosable is that $c'(v_i) = i$, for some vertex $v_i \in V(H_i)$, $1 \leq i \leq 4$.

Then, no matter which color is the coloring $c'(p)$, we will get that the vertex p is adjacent to a vertex in G which has the same color as p, which is not possible. Therefore, we get that graph G is not $\mathfrak{L}'$-choosable and since $|L'(v)| = 4, v \in V(G)$, it follows that the Mirzakhani graph G is not 4-choosable.

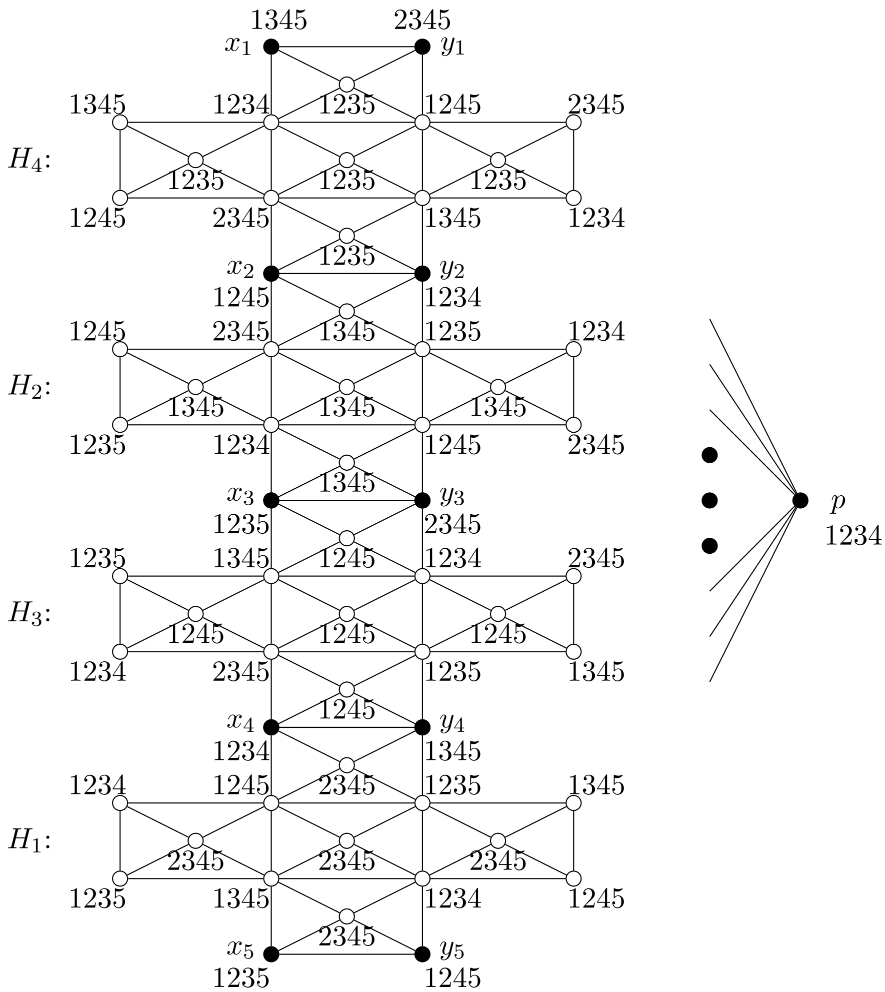


Figure 2.2.5: The Mirzakhani graph:
A non-4-choosable planar graph of order 63

∎

From the fact that for the Mirzakhani graph G it holds that $\chi(G) = 3$, it follows that a 3-colorable planar graph doesn't imply 4-choosability.

2.3 List-edge colorings

The concepts given in Section 2 regarding list colorings have obvious analogues when it comes to edge coloring as we will see in this section. Let $L(e)$ be a list or set of colors for every edge e of a nonempty graph G. Furthermore, let $\mathfrak{L} = \{L(e) : e \in E(G)\}$. We say that the graph G is **$\mathfrak{L}$-edge choosable** or **$\mathfrak{L}$-list edge choosable** is there exists a proper edge coloring c of G such that $c(e) \in L(e)$ for every edge e of G. A graph G is **k-edge-choosable** or **k -list edge colorable** if for every set $\mathfrak{L} = \{L(e) : e \in E(G)\}$, where $|L(e) \geq k|$ for each edge e of G and k is a positive integer, the graph G is $\mathfrak{L}$-edge choosable. The **list chromatic index** $\chi'_L(G)$ is the minimum positive integer k for which G is k-edge choosable. As noted earlier, for any given graph G, it is evident that $\chi_L(G)$ is greater than or equal to the chromatic number $\chi(G)$, $\chi_L(G) \geq \chi(G)$. Similarly, $\chi'_l(G)$ is greater than or equal to the chromatic number $\chi'(G)$,

$$\chi'_l(G) \geq \chi'(G).$$

While the former inequality is strict for certain graphs like $K_{3,3}$, there exists a conjecture that the latter inequality is universally satisfied with equality. The following conjecture, independently proposed by Vadim Vizing, Ram Prakash Gupta, and Michael Albertson and Karen Collins [12], was first published in a 1985 paper by Béla Bollobás and Andrew J. Harris [3].

Conjecture 2.3.1 (The List Edge Coloring Conjecture)**.** *For every loopless graph G it holds that*

$$\chi'_l(G) = \chi'(G).$$

In [7] Jeffrey Howard Dinitz had already conjectured that

$$\chi'_l(G) = \chi'(G),$$

when G is a regular complete bipartite graph. Given that the list chromatic index of a graph is equal to the list chromatic number of its line graph, the Conjecture 2.3.1 can be restated as $\chi'_l(L(G)) = \chi'(L(G))$.

By Theorem 1.2.3, it is known that $\chi'(K_{r,r}) = r$.

Conjecture 2.3.2 (Dinitz's Conjecture)**.** *For every positive integer r,*

$$\chi'_L(K_{r,r}) = r.$$

Conjecture 2.3.1 emerged through the independent contributions of several authors, including V. G. Vizing, R. P. Gupta, as well as M. O. Albertson and K. L. Collins. It made its first appearance in an article authored by Bollobás and Harris in [3], and one can find a succinct historical overview in Häggkvist and Chetwynd in [12]. Fred Galvin, in [9], not only confirmed the validity Conjecture 2.3.2, but also demonstrated that the List Edge Coloring Conjecture, 2.3.1, holds for all bipartite graphs, extending to encompass all bipartite multigraphs as well. Given that coloring a graph's edges amounts to coloring its vertex set of its line graph, the key step of the proof involves establishing that line graphs of bipartite graphs can be directed in a manner that:

(i) maintains a reasonably limited maximum outdegree and

(ii) guarantees the existence of a kernel for every induced subgraph.

2.3.1 Galvin's Lemma

The approach we present to prove this result is built upon the groundwork laid by Tomaž Slivnik, which, in turn, derives from Galvin's original proof. To begin, we invoke a lemma proved by Slivnik.

First, we introduce some notation. Let G be a nonempty bipartite graph, with partitioned sets U and W. For each edge e of G, let u_e denote the vertex of U incident with e, and let w_e denote the corresponding vertex in set W incident with e. When dealing with adjoining edges e and f, it follows that u_e equals u_f or w_e equals w_f, but not both. A **matching** in a graph is a set of pairwise nonadjacent links.

Definition 2.3.1. *Let A be a set of edges of graph G, $M \subseteq A$ be a matching and $c : E(G) \to \mathbb{N}$ be an edge coloring of G. If for every edge $e \in A - M$, there is an edge $f \in M$ such that either:*

(i) $u_f = u_e$ and $c(f) > c(e)$

(ii) $w_f = w_e$ and $c(f) < c(e)$,

then matching M is said to be optimal (in A).

Definition 2.3.2. *An edge $e \in A$ is U-maximum if there is no edge $f \in A$ such that $u_e = u_f$ and $c(f) > c(e)$, while edge $e \in A$ is W-maximum if there is no edge $f \in A$ such that $w_e = w_f$ and $c(f) > c(e)$. An edge $e \in A$ is c-maximum if it is both U-maximum and W-maximum. Hence, an edge $e \in A$ is c-maximum if $c(f) < c(e)$ for every edge f adjacent to e.*

Lemma 2.3.1. *Let $G[U, W]$ be a nonempty bipartite graph and let function $c : E(G) \to \mathbb{N}$ be an edge-coloring of G. For all $e \in E(G)$, let $\sigma_G(e)$ be* *defined as*

$$\sigma_G(e) = 1 + |\{f \in E(G) : u_f = f_e \ \ and \ \ c(f) > c(e)\}| \\ + |\{f \in E(G) : w_f = w_e \ \ and \ \ c(f) < c(e)\}|$$

and let $L(e)$ *be a set of* $\sigma_G(e)$ *colors. If*

$$\mathfrak{L} = \{L(e) : e \in E(G)\},$$

then G *is* $\mathfrak{L}$*-edge-choosable.*

Proof. By induction on number of edges $|E(G)|$.

If $|E(G)| = 1$ the statement is certainly true. Assume the statement of the theorem is true for all nonempty bipartite graphs of size less than m, $m \geq 2$, and let G be a nonempty bipartite graph of size m. Let function c be an edge-coloring defined on G and the numbers $\sigma_G(e)$, the set $L(e)$ and the set $\mathfrak{L}$ are defined in the statement of the theorem.

We will firstly show that for every set of edges $A \subseteq E(G)$, there exists an optimal matching $M \subseteq A$. By induction on the size of A. Note that if A is a matching itself, then $M = A$ is obviously optimal.

If $|A| = 1$ then $M = A$ is optimal and the basis induction step is true.

Now, let us assume that for each set $A' \subseteq E(G)$, such that $|A'| = k - 1$, where $1 < k \leq m$, there exists an optimal matching M' in A'. Let A be a set of edges of G such that $|A| = k$. we will show that there is an optimal matching $M \subseteq A$ considering two cases.

- Case 1. Every U-maximum edge in A is also W-maximum edge in A.

  Let
  $$M = \{e \in A : e \text{ is } c\text{-maximum}\}.$$
  we will prove that M is optimal.

  No two edges of M can be adjacent, so we get that M is a matching. Let $e \in A \setminus M$. Since $e \notin M$, e is not c-maximum, therefore e is not U-maximum. So, there exists an edge $f \in A$ such that $c(f)$ is maximum and $u_e = u_f$. This implies that f is U-maximum and therefore W-maximum, because of assumption, so we get that f is c-maximum, that is, $f \in M$ and $c(f) > c(e)$. So, in this case we get that M is optimal.

- Case 2. There exists an edge $g \in A$ that is U-maximum, but not W-maximum. Since g is not W-maximum, there exists an edge $h \in A$

such that $w_h = w_g$ and $c(h) > c(g)$. Note that $|A \setminus \{h\}| = k - 1$. By induction hypothesis, there exists an optimal matching $M \in A \setminus \{h\}$. Therefore, for every edge e from set $(A \setminus \{h\}) \setminus M = A \setminus (M \cup \{h\})$ there exists an edge $f \in M$ such that either:

(1) $u_f = u_h$ and $c(f) > c(e)$ or (2) $w_e = w_f$ and $c(f) < c(e)$.

To show that $M \in A$ is optimal we first establish the existence of an edge $f \in A$ such that either one of these two cases holds:

(1) $u_f = u_h$ and $c(f) > c(h)$ or (2) $w_f = w_h$ and $c(f) < c(h)$.

Consider two subcases depending on whether edge g is from M or not.

- Subcase 2.1. $g \notin M$. Then $g \in A \setminus (M \cup \{h\})$. By induction hypothesis, there exists an edge $f \in M$ for which one of the following cases is true (1) $u_g = u_f$ and $c(f) > c(g)$ or (2) $w_g = w_f$ and $c(f) < c(g)$. Since edge g is U-maximum case (1) cannot happen, so case (2) must hold. Therefore, we get that $c(f) < c(g) < c(h)$ and M is optimal.
- Subcase 2.2. $g \in M$. Then, $g = f$ and M is optimal.

  Hence, M is optimal in either subcase and, therefore, for every set A of edges of G, there exist an optimal matching $M \subseteq A$.

We can show now that there is an $\mathfrak{L}$-list-edge coloring of graph G. Select an color $a \in \bigcup_{e \in E(G)} L(e)$ and let

$$A = \{e \in E(G) : a \in L(G)\}.$$

Let $G' = G - M$, where M is an optimal matching in A. For each edge $e \in E(G')$, let $L'(e) = L(e) \setminus \{a\}$. If $e \in E(G) \setminus A$, then $a \notin L(e)$ and we have that

$$|L'(e)| = |L(e)| = \sigma_G(e) \geq \sigma_{G'}(e).$$

On the other hand, if $e \in A \setminus M$, then $a \in L(e)$. Since $M \subseteq A$ is optimal, there exists an edge $f \in M$ such that either:
(1) $u_e = u_f$ and $c(f) > c(e)$ or
(2) $w_e = w_f$ and $c(f) < c(e)$.

Therefore,

$$|L'(e)| = |L(e)| - 1 = \sigma_G(e) - 1 \geq \sigma_{G'}(e).$$

Let $\mathfrak{L}' = \{L'(e) : e \in E(G')\}$.

Since $|E(G')| \leq |E(G)|$, by induction hypothesis G' is $\mathfrak{L}'$-edge choosable.

Hence, there exists a proper edge coloring c' of graph G', $c' : E(G') \to \mathbb{N}$, such that $c'(e) \in L'(e)$ for every edge $e \in E(G)$.

Let us define $c : E \to \mathbb{N}$ as

$$c(e) = \begin{cases} c'(e), & e \in E(G) \\ a, & e \in . \end{cases}$$

Then, $c(e) \in L(e)$ for every edge $e \in E(G)$ and $c(e) \neq c(f)$ for every two adjacent edges e and f of G. We get that c is a proper edge coloring of graph G and G is $\mathfrak{L}$-edge-choosable.

■

From Lemma 2.3.1, we can now present a proof of Galvin's theorem

Theorem 2.3.1. (Galvin′s Theorem) *If G is a bipartite graph, then*

$$\chi'_L(G) = \chi'(G).$$

Proof. By König's Theorem 1.2.3, since G is bipartite, we get that G is of Class one, that is, $\chi'_L(G) = \Delta(G) = \Delta$. Hence, there exists a proper edge-coloring $c : E(G) \to \{1, 2, ..., \Delta\}$ of G.

Let $L(e)$ be a list of colors such that for each edge $e \in E(G)$ it holds

$$\begin{aligned} |L(e)| = \sigma_G(e) = 1 &+ |\{f \in E(G) : u_e = u_f \ \ and \ \ c(f) > c(e)\}| \\ &+ |\{f \in E(G) : w_e = w_f \ \ and \ \ c(f) < c(e)\}| \\ &\leq 1 + (\chi'(G) - c(e)) + (c(e) - 1) = \chi'(G). \end{aligned}$$

Let $\mathfrak{L} = \{L(e) : e \in E(G)\}$. By Lemma 2.3.1, G is $\mathfrak{L}$-choosable. Hence, G is $\chi'(G)$-edge choosable, so $\chi'_L(G) \leq \chi'(G)$.

Combining this result with the fact that $\chi'(G) \leq \chi'_L(G)$ we get that $\chi'_L(G) = \chi'(G)$.

■

Since every bipartite graph is of Class one, it follows that the list chromatic index of every bipartite graph G equals $\Delta(G)$.

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Kratka biografija

Marija Jovanović je rođena 26. decembra 1995. godine u Nišu. Odrasla je u svom rodnom gradu, gde je 2010. godine završila Osnovnu školu "Dušan Radović" kao vukovac, a 2014. godine i Gimnaziju "Bora Stanković", takođe kao vukovac. Nakon završene srednje škole upisuje studije matematike na Departmanu za matematiku Prirodno-matematičkog fakulteta Univerziteta u Nišu, 2015. godine. Osnovne studije završava 2018. godine nakon čega upisuje i master studije matematike, modul primenjena matematika - nauka o podacima, na Departmanu za matematiku i informatiku Prirodno-matematičkog fakulteta Univerziteta u Novom Sadu . Zaključno sa februarom 2022. položila je sve ispite predviđene planom i programom studija.

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